

What do homotopy algebras form?

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Abstract

In paper [4], we constructed a symmetric monoidal category $\mathfrak{SLie}_\infty^{\text{MC}}$ whose objects are shifted (and filtered) L_∞ -algebras. Here, we fix a cooperad \mathcal{C} and show that algebras over the operad $\text{Cobar}(\mathcal{C})$ naturally form a category enriched over $\mathfrak{SLie}_\infty^{\text{MC}}$. Following [4], we “integrate” this $\mathfrak{SLie}_\infty^{\text{MC}}$ -enriched category to a simplicial category $\text{HoAlg}_\mathcal{C}^\Delta$ whose mapping spaces are Kan complexes. The simplicial category $\text{HoAlg}_\mathcal{C}^\Delta$ gives us a particularly nice model of an $(\infty, 1)$ -category of $\text{Cobar}(\mathcal{C})$ -algebras. We show that the homotopy category of $\text{HoAlg}_\mathcal{C}^\Delta$ is the localization of the category of $\text{Cobar}(\mathcal{C})$ -algebras and ∞ -morphisms with respect to ∞ quasi-isomorphisms. Finally, we show that the Homotopy Transfer Theorem is a simple consequence of the Goldman-Millson theorem.

1 Introduction

This work is motivated by Dmitry Tamarkin’s answer [24] to Vladimir Drinfeld’s question “What do dg categories form?” [10], and by papers [1], [8], [13], [15], and [18]. Here, we give an answer to the question “What do homotopy algebras form?”

Homotopy algebras and their generalizations appear in constructions of string topology, in rational homotopy theory, symplectic topology, deformation quantization, and quantum field theory. For a gentle introduction to this topic, we refer the reader to paper [25]. For a more detailed exposition, a standard reference is book [19] by Jean-Louis Loday and Bruno Vallette.

Despite an important role of homotopy algebras, it is not clear what higher categorical structure stands behind the homotopy theory of homotopy algebras. A possible way to answer to this question is to use closed model categories and this approach is undertaken in paper [26] by Bruno Vallette. Here, we suggest a different approach which is based on the use of the convolution L_∞ -algebra and the Getzler-Hinich construction [13], [15].

By a homotopy algebra structure on a cochain complex of \mathbb{k} -vector¹ spaces A we mean a $\text{Cobar}(\mathcal{C})$ -algebra structure on A , where \mathcal{C} is a differential graded (dg) cooperad satisfying some technical conditions. In this paper, we fix such a cooperad \mathcal{C} and show that $\text{Cobar}(\mathcal{C})$ -algebras form a $\mathfrak{SLie}_\infty^{\text{MC}}$ -enriched category $\text{HoAlg}_\mathcal{C}$, where $\mathfrak{SLie}_\infty^{\text{MC}}$ is the enhancement of the symmetric monoidal category of shifted L_∞ -algebras introduced in² [4]. Then, using a generalization of the Getzler-Hinich construction [13], [15], we show that the $\mathfrak{SLie}_\infty^{\text{MC}}$ -category of $\text{Cobar}(\mathcal{C})$ -algebras can be “integrated” to the category $\text{HoAlg}_\mathcal{C}^\Delta$ enriched over ∞ -groupoids (a.k.a. Kan complexes).

¹In this paper, we assume that $\text{char}(\mathbb{k}) = 0$.

²See also Section 3.1 of this paper.

Our approach is conceptually similar to a standard construction of the simplicial category of chain complexes i.e., “trivial” homotopy algebras. There, instead of constructing the simplicial mapping spaces directly, one exploits the simple fact that chain complexes are naturally enriched over chain complexes i.e., abelian L_∞ -algebras. The simplicial category is then constructed by first truncating the mapping complexes and applying the Dold-Kan functor. It has been noted, for example by E. Getzler in [13], that the integration of a L_∞ -algebra is, up to homotopy, a non-abelian analogue of the Dold-Kan functor. Hence, to construct the simplicial category $\mathbf{HoAlg}_\mathcal{C}^\Delta$ of non-trivial homotopy algebras, we proceed via analogy by first considering a category of homotopy algebras enriched over L_∞ -algebras i.e., “non-abelian complexes”.

Furthermore, we prove that the homotopy category of $\mathbf{HoAlg}_\mathcal{C}^\Delta$ is the localization of the category of $\mathbf{Cobar}(\mathcal{C})$ -algebras with respect to ∞ quasi-isomorphisms, and this is indeed the correct homotopy category of homotopy algebras (see for example [19, Thm. 11.4.12]). Thus the simplicial category $\mathbf{HoAlg}_\mathcal{C}^\Delta$ is the sought higher categorical structure which stands behind the homotopy category of homotopy algebras.

For simplicity of the exposition, we present all constructions in the “1-colored” setting. However, it is not hard to see the all the statements can be easily extended to the multi-colored setting.

Our paper is organized as follows. Section 2 is devoted to prerequisites on homotopy algebras. In this section, we describe the convolution L_∞ -algebra $\mathrm{Conv}(V, A)$ associated to a \mathcal{C} -coalgebra V and a $\mathbf{Cobar}(\mathcal{C})$ -algebra A , and recall [8] that, for every pair of $\mathbf{Cobar}(\mathcal{C})$ -algebras A, B , Maurer-Cartan (MC) elements of $\mathrm{Conv}(\mathcal{C}(A), B)$ are in bijection with ∞ -morphisms from A to B . Section 3 starts with a brief reminder of the symmetric monoidal category $\mathfrak{SLie}_\infty^{\mathrm{MC}}$ introduced in [4]. Then we construct the $\mathfrak{SLie}_\infty^{\mathrm{MC}}$ -enriched category $\mathbf{HoAlg}_\mathcal{C}$ and the corresponding simplicial category $\mathbf{HoAlg}_\mathcal{C}^\Delta$. In Section 4, we show that $\pi_0(\mathbf{HoAlg}_\mathcal{C}^\Delta)$ is the homotopy category of $\mathbf{Cobar}(\mathcal{C})$ -algebras, i.e. every ∞ quasi-isomorphism of $\mathbf{Cobar}(\mathcal{C})$ -algebras gives an invertible morphism in $\pi_0(\mathbf{HoAlg}_\mathcal{C}^\Delta)$ and the category $\pi_0(\mathbf{HoAlg}_\mathcal{C}^\Delta)$ has the desired universal property. Also in Section 4, we characterize ∞ quasi-isomorphisms as precisely those morphisms which induce homotopy equivalences between mapping spaces via (pre)composition. Finally, in Section 5, we give a very concise proof of the Homotopy Transfer Theorem for homotopy algebras. This proof is based on a construction from [7] and a version of the Goldman-Millson theorem from [2].

On a more general definition of a homotopy algebra. One could also define a homotopy algebra as an algebra over an operad \mathcal{O} which is freely generated by a collection \mathcal{C} when viewed as an operad in the category of graded vector spaces, where the collection \mathcal{C} and the differential on \mathcal{O} are subject to some technical conditions. It is not hard to generalize all the constructions of our paper to this more general setting. However, for simplicity of the exposition, we decided to present the whole story for the case when \mathcal{O} is the cobar construction applied to a fixed cooperad.

Is the $\mathfrak{SLie}_\infty^{\mathrm{MC}}$ -enriched category of L_∞ -algebras related to the rational homotopy theory? It is not surprising that the answer to this question is “yes”. However, a precise formulation of the answer requires some technical conditions on L_∞ -algebras and some

amendments to the definition of the mapping space. So separate note [5] is devoted to this question.

Related work on this subject

While this preprint was in preparation, paper [16] appeared on arXiv.org. In this paper, the authors constructed a quasi-category of Leibniz ∞ -algebras also using the Getzler-Hinich construction [13], [15].

Our paper agrees in spirit with treatise [20] by Jacob Lurie. In this treatise, J. Lurie develops a systematic approach to algebraic structures on objects in a fixed symmetric monoidal ∞ -category. This approach allows him to define E_∞ -ring as a commutative algebra object in the ∞ -category of spectra. Even though, the framework of the presentation of [20] is very general, we could not find a particular statement in the current version of draft [20] from which all statements proved in our paper will follow. One of the reasons for this is that, in [20], J. Lurie usually assumes the all the (co)chain complexes under consideration satisfy some kind of boundedness condition whereas in our paper we do not impose this assumption.

In paper [26], Bruno Vallette used the simplicial localization methods of Dwyer and Kan [11] to upgrade the category of Cobar(\mathcal{C})-algebras with ∞ -morphisms (for a cooperad \mathcal{C} satisfying some conditions) to a simplicial category (see Theorem 3.5 in [26]). The homotopy category of this simplicial category is isomorphic to that of $\mathbf{HoAlg}_{\mathcal{C}}^\Delta$ and we believe that there is a more precise link between these two simplicial categories.

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Notation and conventions

A big part of our conventions is borrowed from [4]. For example, the ground field \mathbb{k} has characteristic zero and $\mathbf{Ch}_{\mathbb{k}}$ denotes the category of unbounded cochain complexes of \mathbb{k} -vector spaces. Any \mathbb{Z} -graded vector space V is tacitly considered as the cochain complex with the zero differential. Following [4] we frequently use the ubiquitous abbreviation “dg” (differential graded) to refer to algebraic objects in $\mathbf{Ch}_{\mathbb{k}}$. The notation $\mathbf{s}V$ (resp. by $\mathbf{s}^{-1}V$) is reserved for the suspension (resp. the desuspension) of a cochain complex V , i.e. $(\mathbf{s}V)^\bullet = V^{\bullet-1}$ and $(\mathbf{s}^{-1}V)^\bullet = V^{\bullet+1}$. Finally, for a pair V, W of \mathbb{Z} -graded vector spaces we denote by

$$\mathrm{Hom}(V, W)$$

the corresponding inner-hom object in the category of \mathbb{Z} -graded vector spaces.

Following [4] we tacitly assume the Koszul sign rule. In particular,

$$(-1)^{\varepsilon(\sigma; v_1, \dots, v_m)}$$

will always denote the sign factor corresponding to the permutation $\sigma \in S_m$ of homogeneous vectors v_1, v_2, \dots, v_m . Namely,

$$(-1)^{\varepsilon(\sigma; v_1, \dots, v_m)} := \prod_{(i < j)} (-1)^{|v_i||v_j|}, \quad (1.1)$$

where the product is taken over all inversions $(i < j)$ of $\sigma \in S_m$.

For a finite group G acting on a cochain complex (or a graded vector space) V , we denote by

$$V^G, \quad \text{and} \quad V_G,$$

respectively, the subcomplex of G -invariants in V and the quotient complex of G -coinvariants. Using the advantage of the zero characteristic, we often identify V_G with V^G via this isomorphism

$$v \mapsto \sum_{g \in G} g(v) : V_G \rightarrow V^G. \quad (1.2)$$

For a graded vector space (or a cochain complex) V the notation $S(V)$ (resp. $\underline{S}(V)$) is reserved for the underlying vector space of the symmetric algebra (resp. the truncated symmetric algebra) of V :

$$S(V) = \mathbb{k} \oplus V \oplus S^2(V) \oplus S^3(V) \oplus \dots,$$

$$\underline{S}(V) = V \oplus S^2(V) \oplus S^3(V) \oplus \dots,$$

where

$$S^n(V) = (V^{\otimes_{\mathbb{k}} n})_{S_n}.$$

We denote by **As**, **Com**, **Lie** the operads governing associative, commutative (and associative), and Lie algebras, respectively. We set $\mathbf{As}(0) = \mathbf{Com}(0) = \mathbf{0}$. In other words, algebras over **As** and **Com** are non-unital. We denote by \mathbf{As}_∞ , \mathbf{Com}_∞ , and \mathbf{Lie}_∞ the dg operads which govern homotopy versions of the corresponding algebras. Furthermore, we denote by **coAs**, **coCom**, and **coLie**, the cooperads which are obtained from **As**, **Com**, and **Lie** respectively, by taking the linear dual.

For an augmented operad \mathcal{O} , we denote by \mathcal{O}_\circ the kernel of the augmentation. Dually, for a coaugmented cooperad \mathcal{C} we denote by \mathcal{C}_\circ the cokernel of the coaugmentation. Recall that for every augmented operad \mathcal{O} (resp. coaugmented cooperad \mathcal{C}) the collection \mathcal{O}_\circ (resp. \mathcal{C}_\circ) is naturally a pseudo-operad (resp. pseudo-cooperad) in the sense of [3, Section 3.2] (resp. [3, Section 3.4]).

For an operad (resp. a cooperad) P and a cochain complex V we denote by $P(V)$ the free P -algebra (resp. the cofree³ P -coalgebra) generated by V :

$$P(V) := \bigoplus_{n \geq 0} (P(n) \otimes V^{\otimes n})_{S_n}. \quad (1.3)$$

For example,

$$\mathbf{Com}(V) = \underline{S}(V) \quad \text{and} \quad \mathbf{coCom}(V) = \underline{S}(V).$$

³In this paper we only consider nilpotent coalgebras.

For a cooperad \mathcal{C} , it is sometimes more convenient to work with the cofree \mathcal{C} -coalgebra defined as the direct sum of the space of invariants (instead of coinvariants):

$$\bigoplus_{n \geq 0} \left(\mathcal{C}(n) \otimes V^{\otimes n} \right)^{S_n}. \quad (1.4)$$

For example, it is more natural to define a \mathcal{C} -coalgebra structure on a graded vector space (or a cochain complex) V as a collection of comultiplication maps

$$\Delta_n : V \rightarrow \left(\mathcal{C}(n) \otimes V^{\otimes n} \right)^{S_n} \quad (1.5)$$

satisfying some natural coassociativity axioms which are obtained by dualizing the corresponding axioms for algebras over an operad.

We denote this direct sum by

$$\mathcal{C}(V)^{\text{inv}} := \bigoplus_{n \geq 0} \left(\mathcal{C}(n) \otimes V^{\otimes n} \right)^{S_n} \quad (1.6)$$

and keep in mind that $\mathcal{C}(V)^{\text{inv}}$ is isomorphic to $\mathcal{C}(V)$ via map (1.2).

Following [4], \mathfrak{S} and \mathfrak{S}^{-1} denote the underlying collections of the operads

$$\text{End}_{\mathfrak{s}^{-1}\mathbb{k}}, \quad \text{and} \quad \text{End}_{\mathfrak{s}\mathbb{k}},$$

respectively. Furthermore, for a dg (co)operad P , we denote by $\mathfrak{S}P$ (resp. $\mathfrak{S}^{-1}P$) the dg (co)operad which is obtained from P by tensoring with \mathfrak{S} (resp. \mathfrak{S}^{-1}):

$$\mathfrak{S}P := \mathfrak{S} \otimes P, \quad \mathfrak{S}^{-1}P := \mathfrak{S}^{-1} \otimes P.$$

For example, $\mathfrak{S}\text{Lie}_\infty$ -algebras are algebra over the dg operad

$$\mathfrak{S}\text{Lie}_\infty := \text{Cobar}(\text{coCom}) \quad (1.7)$$

and L_∞ -algebras are algebras over the dg operad

$$\text{Lie}_\infty := \text{Cobar}(\mathfrak{S}^{-1}\text{coCom}). \quad (1.8)$$

Just as in [4], we often call $\mathfrak{S}\text{Lie}_\infty$ -algebras *shifted L_∞ -algebras*. Although a $\mathfrak{S}\text{Lie}_\infty$ -algebra structure on a cochain complex V is the same thing as an L_∞ structure on $\mathfrak{s}V$, working with $\mathfrak{S}\text{Lie}_\infty$ -algebras has important technical advantages. This is why we prefer to deal with shifted L_∞ structures on V versus original L_∞ structures on $\mathfrak{s}V$.

Following [4], we denote the tensor product of ∞ -morphisms of $\mathfrak{S}\text{Lie}_\infty$ -algebras by \otimes even though the tensor product of the $\mathfrak{S}\text{Lie}_\infty$ -algebras is denoted by \oplus .

We often use the plain arrow \rightarrow for ∞ -morphisms of homotopy algebras. Of course, it should be kept in mind that in general such morphisms are maps of the corresponding coalgebras but not the underlying cochain complexes.

The abbreviation “MC” is reserved for the term “Maurer-Cartan”.

Conventions about trees. By a *tree* we mean a connected graph without cycles with a marked vertex called *the root*. In this paper, we assume that the root of every tree has valency 1 (such trees are sometimes called *planted*). The edge adjacent to the root is called the *root edge*. Non-root vertices of valency 1 are called *leaves*. A vertex is called *internal* if it is neither a root nor a leaf. We always orient trees in the direction towards the root. Thus every internal vertex has at least 1 incoming edge and exactly 1 outgoing edge. An edge adjacent to a leaf is called *external*. A tree \mathbf{t} is called *planar* if, for every internal vertex v of \mathbf{t} , the set of edges terminating at v carries a total order.

Let us recall [3, Section 2] that for every planar tree \mathbf{t} the set $V(\mathbf{t})$ of all its vertices is equipped with a natural total order such that the root is the smallest vertex of the tree.

For a non-negative integer n , an n -labeled planar tree \mathbf{t} is a planar tree equipped with an injective map

$$\mathfrak{l} : \{1, 2, \dots, n\} \rightarrow L(\mathbf{t}) \quad (1.9)$$

from the set $\{1, 2, \dots, n\}$ to the set $L(\mathbf{t})$ of leaves of \mathbf{t} . Although the set $L(\mathbf{t})$ has a natural total order we do not require that map (1.9) is monotonous.

The set $L(\mathbf{t})$ of leaves of an n -labeled planar tree \mathbf{t} splits into the disjoint union of the image $\mathfrak{l}(\{1, 2, \dots, n\})$ and its complement. We call leaves in the image of \mathfrak{l} *labeled*.

A vertex x of an n -labeled planar tree \mathbf{t} is called *nodal* if it is neither the root, nor a labeled leaf. We denote by $V_{\text{nod}}(\mathbf{t})$ the set of all nodal vertices of \mathbf{t} . Keeping in mind the canonical total order on the set of all vertices of \mathbf{t} we can say things like “the first nodal vertex”, “the second nodal vertex”, and “the i -th nodal vertex”.

It is convenient to talk about (co)operads and pseudo-(co)operads using the groupoid $\text{Tree}(n)$ of n -labeled planar trees. Objects of $\text{Tree}(n)$ are n -labeled planar trees and morphisms are non-planar isomorphisms of the corresponding (non-planar) trees compatible with labeling.

Following [3, Section 3.2, 3.4], for an n -labelled planar tree \mathbf{t} and pseudo-operad P (resp. pseudo-cooperad Q) the notation $\mu_{\mathbf{t}}$ (resp. the notation $\Delta_{\mathbf{t}}$) is reserved for the multiplication map

$$\mu_{\mathbf{t}} : P(r_1) \otimes P(r_2) \otimes \dots \otimes P(r_k) \rightarrow P(n) \quad (1.10)$$

and the comultiplication map

$$\Delta_{\mathbf{t}} : Q(n) \rightarrow Q(r_1) \otimes Q(r_2) \otimes \dots \otimes Q(r_k). \quad (1.11)$$

respectively. Here, k is the number of nodal vertices of the planar tree \mathbf{t} and r_i is the number of edges (of \mathbf{t}) which terminate at the i -th nodal vertex of \mathbf{t} .

For example, if $\mathbf{t}_{n,k,i}$ is the labeled planar tree shown on figure⁴ 1 then the map

$$\mu_{\mathbf{t}_{n,k,i}} : P(n) \otimes P(k) \rightarrow P(n + k - 1) \quad (1.12)$$

is precisely the i -th elementary insertion

$$\circ_i : P(n) \otimes P(k) \rightarrow P(n + k - 1). \quad (1.13)$$

⁴On figures, small white circles are used for nodal vertices and small black circles are used for all the remaining vertices.

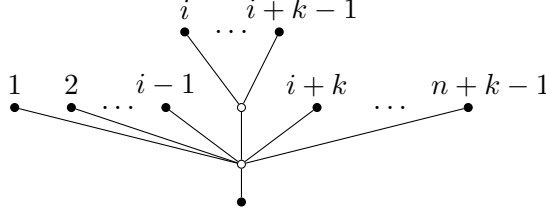


Figure 1: The $(n+k-1)$ -labeled planar tree $\mathbf{t}_{n,k,i}$

Recall that a (co)operad P is *reduced* if it satisfies this technical condition

$$P(0) = \mathbf{0}. \quad (1.14)$$

When we deal with reduced (co)operads, we may discard all labeled trees which have at least one nodal vertex with no incoming edges. In other words, one may safely assume that nodal vertices of a labeled tree are precisely its internal vertices, i.e. map (1.9) is a bijection.

2 Prerequisites on homotopy algebras

Let \mathcal{C} be a dg coaugmented cooperad satisfying the following technical condition:

Condition 2.1 *The pseudo-cooperad \mathcal{C}_\circ carries an ascending filtration*

$$\mathbf{0} = \mathcal{F}^0 \mathcal{C}_\circ \subset \mathcal{F}^1 \mathcal{C}_\circ \subset \mathcal{F}^2 \mathcal{C}_\circ \subset \mathcal{F}^3 \mathcal{C}_\circ \subset \dots \quad (2.1)$$

which is compatible with the pseudo-cooperad structure in the following sense:

$$\Delta_{\mathbf{t}}(\mathcal{F}^m \mathcal{C}_\circ(n)) \subset \bigoplus_{m_1 + \dots + m_k = m} \mathcal{F}^{m_1} \mathcal{C}_\circ(r_1) \otimes \mathcal{F}^{m_2} \mathcal{C}_\circ(r_2) \otimes \dots \otimes \mathcal{F}^{m_k} \mathcal{C}_\circ(r_k), \quad (2.2)$$

$$\mathbf{t} \in \text{Tree}(n),$$

where k is the number of nodal vertices of the planar tree \mathbf{t} and r_i is the number of edges (of \mathbf{t}) which terminate at the i -th nodal vertex of \mathbf{t} . We also assume that \mathcal{C}_\circ is cocomplete with respect to filtration (2.1), i.e.

$$\mathcal{C}_\circ = \bigcup_m \mathcal{F}^m \mathcal{C}_\circ. \quad (2.3)$$

We will use the following pedestrian definition of homotopy algebras of a given type:

Definition 2.2 *Homotopy algebras of type \mathcal{C} are algebras in $\mathbf{Ch}_{\mathbb{k}}$ over the operad $\text{Cobar}(\mathcal{C})$.*

Thus, A_∞ -, L_∞ -, and Com_∞ -algebras are examples of homotopy algebras. Indeed, A_∞ -algebras are algebras over the operad $\text{Cobar}(\mathfrak{S}^{-1}\text{coAs})$, L_∞ -algebras are algebras over $\text{Cobar}(\mathfrak{S}^{-1}\text{coCom})$, and Com_∞ -algebras are algebras over $\text{Cobar}(\mathfrak{S}^{-1}\text{coLie})$.

Let A be a cochain complex and let

$$\text{coDer}(\mathcal{C}(A)) \quad (2.4)$$

be the dg Lie algebra of coderivations of the \mathcal{C} -coalgebra $\mathcal{C}(A)$.

It is known [3, Corollary 5.3], [14, Proposition 2.15], that $\text{Cobar}(\mathcal{C})$ -algebra structures on A are in bijection with MC elements of the dg Lie subalgebra

$$\text{coDer}'(\mathcal{C}(A)) \subset \text{coDer}(\mathcal{C}(A)) \quad (2.5)$$

of coderivations Q satisfying the condition

$$Q|_A = 0. \quad (2.6)$$

Thus every homotopy algebra A of type \mathcal{C} gives us a (dg) \mathcal{C} -coalgebra

$$(\mathcal{C}(A), \partial + Q) \quad (2.7)$$

where ∂ is the differential on $\mathcal{C}(A)$ induced by the ones on A and \mathcal{C} .

This observation is used to define a notion of ∞ -morphism of homotopy algebras of type \mathcal{C} . In other words,

Definition 2.3 *Let A and B be homotopy algebras of type \mathcal{C} and let Q_A (resp. Q_B) be the MC element of $\text{coDer}(\mathcal{C}(A))$ (resp. $\text{coDer}(\mathcal{C}(B))$) corresponding to the homotopy algebra structure on A (resp. on B). Then, an ∞ -morphism from A to B is a homomorphism of dg \mathcal{C} -coalgebras*

$$U : (\mathcal{C}(A), \partial + Q_A) \rightarrow (\mathcal{C}(B), \partial + Q_B). \quad (2.8)$$

Recall that any such homomorphism U is uniquely determined by its composition

$$U' := p_B \circ U \quad (2.9)$$

with the canonical projection $p_B : \mathcal{C}(B) \rightarrow B$. In this paper, we often use this convention: U' denotes composition (2.9) corresponding to a homomorphism of \mathcal{C} -coalgebras U (2.8).

Given an ∞ -morphism U from A_1 to A_2 and an ∞ -morphism \tilde{U} from A_2 to A_3 , their composition is defined, in the obvious way, as the composition of the corresponding homomorphisms of dg \mathcal{C} -coalgebras. We denote by

$$\text{Cat}_{\mathcal{C}}$$

the category whose objects are homotopy algebras of type \mathcal{C} (a.k.a. $\text{Cobar}(\mathcal{C})$ -algebras) and whose morphisms are ∞ -morphisms with the above obvious composition.

Remark 2.4 Due to [23, Proposition 38], Condition 2.1 implies that $\text{Cobar}(\mathcal{C})$ is a cofibrant object in the closed model category of dg operads. The same condition also guarantees that homotopy algebras of type \mathcal{C} enjoy the obvious version of the Homotopy Transfer Theorem (see [19, Theorem 10.3.2]). A very concise proof of this important theorem is given in Section 5 of this paper.

Remark 2.5 Another way to define the notion of ∞ -morphism of homotopy algebras is to resolve a 2-colored operad which governs pairs of algebras with a morphism between them. This different approach to the “higher category” of homotopy algebras is initiated in works [9], [21] of M. Dubeck and M. Markl.

2.1 The convolution $\mathfrak{S}\text{Lie}_\infty$ -algebra

Let V be a \mathcal{C} -coalgebra and A be a homotopy algebra of type \mathcal{C} (i.e. an algebra over $\text{Cobar}(\mathcal{C})$).

On the graded vector space

$$\text{Hom}(V, A) \quad (2.10)$$

we define the following multi-brackets:

$$\{f\}(v) = \partial_A f(v) - (-1)^{|f|} f(\partial_V v) + p_A \circ Q_A(1 \otimes f(\Delta_1(v))) \quad (2.11)$$

$$\{f_1, \dots, f_m\}(v) = p_A \circ Q_A(1 \otimes f_1 \otimes \dots \otimes f_m(\Delta_m(v))), \quad m \geq 2, \quad (2.12)$$

where Δ_m is the m -th component of the comultiplication

$$\Delta_m : V \rightarrow (\mathcal{C}(m) \otimes V^{\otimes m})^{S_m}$$

and p_A is the canonical projection

$$p_A : \mathcal{C}(A) \rightarrow A.$$

Note that, since Q_A has degree 1, each multi-bracket in (2.12) also carries degree 1.

We claim that

Proposition 2.6 *For every \mathcal{C} -coalgebra V and a $\text{Cobar}(\mathcal{C})$ -algebra A , multi-brackets (2.11), (2.12) equip the graded vector space $\text{Hom}(V, A)$ with a structure of a $\mathfrak{S}\text{Lie}_\infty$ -algebra.*

The proof of this proposition is given in Appendix A.

Definition 2.7 *Let V be a \mathcal{C} -coalgebra and A be a homotopy algebra of type \mathcal{C} . Then $\mathfrak{S}\text{Lie}_\infty$ -algebra (2.10) is called the convolution algebra of the pair (V, A) . We use the notation:*

$$\text{Conv}(V, A) := \text{Hom}(V, A).$$

2.1.1 Convolution $\mathfrak{S}\text{Lie}_\infty$ -algebra and ∞ -morphisms

For a pair A, B of homotopy algebras of type \mathcal{C} , we consider the convolution $\mathfrak{S}\text{Lie}_\infty$ -algebra

$$L = \text{Hom}(\mathcal{C}(A), B), \quad (2.13)$$

where the \mathcal{C} -coalgebra $\mathcal{C}(A)$ is considered with the differential $\partial + Q_A$, and ∂ comes from the differential on A and the differential on \mathcal{C} .

We observe that the $\mathfrak{S}\text{Lie}_\infty$ -algebra carries the following descending filtration

$$\begin{aligned} \mathcal{F}_0^{\text{ari}} L &\supset \mathcal{F}_1^{\text{ari}} L \supset \mathcal{F}_2^{\text{ari}} L \supset \dots \\ \mathcal{F}_n^{\text{ari}} L &= \{f \in \text{Hom}(\mathcal{C}(A), B) \mid f|_{\mathcal{C}(m) \otimes_{S_m} A^{\otimes m}} = 0 \ \forall m < n\}. \end{aligned} \quad (2.14)$$

It is also easy to check that:

Proposition 2.8 *The convolution \mathfrak{SLie}_∞ -algebra structure given by (2.11) and (2.12) is compatible with filtration (2.14) i.e.*

$$\left\{ \mathcal{F}_{i_1}^{\text{ari}} L, \mathcal{F}_{i_2}^{\text{ari}} L, \dots, \mathcal{F}_{i_k}^{\text{ari}} L \right\} \subseteq \mathcal{F}_{i_1+i_2+\dots+i_k}^{\text{ari}} L \quad \forall \quad k > 1,$$

Moreover, the \mathfrak{SLie}_∞ -algebra $L = \text{Hom}(\mathcal{C}(A), B)$ is complete with respect to this filtration, i.e.

$$L = \varprojlim_k L / \mathcal{F}_k^{\text{ari}} L.$$

□

The notion of the convolution \mathfrak{SLie}_∞ -algebra is partially justified by the following lemma:

Lemma 2.9 *Let A and B be homotopy algebras of type \mathcal{C} . If the cooperad \mathcal{C} satisfies condition*

$$\mathcal{C}(0) = \mathbf{0} \tag{2.15}$$

then

$$\text{Hom}(\mathcal{C}(A), B) = \mathcal{F}_1^{\text{ari}} \text{Hom}(\mathcal{C}(A), B). \tag{2.16}$$

In particular, the \mathfrak{SLie}_∞ -algebra $\text{Hom}(\mathcal{C}(A), B)$ is pro-nilpotent. Furthermore, the assignment

$$U \mapsto U' := p_B \circ U$$

is a bijection between the set of ∞ -morphisms from A to B and the set of MC elements of the \mathfrak{SLie}_∞ -algebra $\text{Hom}(\mathcal{C}(A), B)$.

Proof. The first statement of the lemma follows directly from the definition of filtration (2.14) on the \mathfrak{SLie}_∞ -algebra $\text{Hom}(\mathcal{C}(A), B)$. Since $\text{Hom}(\mathcal{C}(A), B)$ is complete with respect to this filtration, we conclude that the \mathfrak{SLie}_∞ -algebra $\text{Hom}(\mathcal{C}(A), B)$ is pronilpotent.

This conclusion allows us to write the MC equation in the \mathfrak{SLie}_∞ -algebra $\text{Hom}(\mathcal{C}(A), B)$ for any degree zero element.

According to Definition 2.3, an ∞ -morphism from A to B is a homomorphism of dg \mathcal{C} -coalgebras

$$U : (\mathcal{C}(A), \partial + Q_A) \rightarrow (\mathcal{C}(B), \partial + Q_B).$$

Since the \mathcal{C} -coalgebra $\mathcal{C}(B)$ is cofree (over graded vector spaces), the homomorphism U is uniquely determined by its composition

$$U' := p_B \circ U : \mathcal{C}(A) \rightarrow B. \tag{2.17}$$

Furthermore, the compatibility of U with the differentials $\partial + Q_A$ and $\partial + Q_B$ is equivalent to the equation

$$\begin{aligned} & \partial \circ U'(X; a_1, \dots, a_m) \\ & + Q_B \circ (1 \otimes U') \circ \Delta_1(X; a_1, \dots, a_m) - U' \circ (\partial + Q_A)(X; a_1, \dots, a_m) \\ & + \sum_{k=2}^{\infty} \frac{1}{k!} Q_B \circ (1 \otimes (U')^{\otimes k}) \circ \Delta_k(X; a_1, \dots, a_m) = 0, \end{aligned} \tag{2.18}$$

where $(X; a_1, \dots, a_m)$ represents a vector in $\mathcal{C}(A)$ and the factor $1/k!$ in the last sum comes from the identification of $\mathcal{C}(B)^{\text{inv}}$ with $\mathcal{C}(B)$ via the inverse of isomorphism (1.2).

Using the definition of multi-brackets (2.11), (2.12) on $\text{Hom}(\mathcal{C}(A), B)$, we see that (2.18) is precisely the MC equation for U' in the $\mathfrak{S}\text{Lie}_\infty$ -algebra $\text{Hom}(\mathcal{C}(A), B)$. Thus

$$U \leftrightarrow U' := p_B \circ U$$

is a desired bijection between the set of ∞ -morphisms from A to B and the set of MC elements of the $\mathfrak{S}\text{Lie}_\infty$ -algebra $\text{Hom}(\mathcal{C}(A), B)$. \square

Remark 2.10 A version of Lemma 2.9 is proved in [8]. See Proposition 3 in [8, Section 1.3].

Example 2.11 Let us recall that coAs is the cooperad which governs coassociative coalgebras without counit and the dg operad $\text{Cobar}(\mathfrak{S}^{-1}\text{coAs})$ governs (flat) A_∞ -algebras. It is easy to see that, for every A_∞ -algebra A ,

$$\text{Hom}(\mathfrak{S}^{-1}\text{coAs}(A), A) \tag{2.19}$$

is the completed version of the truncated Hochschild cochain complex of A

$$\prod_{n \geq 1} s^{n-1} \text{Hom}(A^{\otimes n}, A) \tag{2.20}$$

and the shifted L_∞ -algebra on (2.20) is obtain by symmetrizing the cup product and its higher analogues. It is worthy of mentioning that the Hochschild differential on (2.20) is obtained by twisting the differential on (2.19) by the MC element corresponding to the identity map $\text{id} : A \rightarrow A$.

3 The $\mathfrak{S}\text{Lie}_\infty^{\text{MC}}$ -enriched category $\text{HoAlg}_{\mathcal{C}}$ and the simplicial category $\text{HoAlg}_{\mathcal{C}}^\Delta$

3.1 A brief reminder of the symmetric monoidal category $\mathfrak{S}\text{Lie}_\infty^{\text{MC}}$

Let us recall [4] that a $\mathfrak{S}\text{Lie}_\infty$ -algebra $(L, \partial, \{\cdot, \cdot\}, \{\cdot, \cdot, \cdot\}, \dots)$ is *filtered* if the underlying complex (L, ∂) is equipped with a complete descending filtration,

$$L = \mathcal{F}_1 L \supset \mathcal{F}_2 L \supset \mathcal{F}_3 L \dots \tag{3.1}$$

$$L = \varprojlim_k L / \mathcal{F}_k L, \tag{3.2}$$

which is compatible with the brackets, i.e.

$$\left\{ \mathcal{F}_{i_1} L, \mathcal{F}_{i_2} L, \dots, \mathcal{F}_{i_m} L \right\} \subseteq \mathcal{F}_{i_1 + i_2 + \dots + i_m} L \quad \forall \quad m > 1.$$

For example, if A and B are $\text{Cobar}(\mathcal{C})$ -algebras and the cooperad \mathcal{C} satisfies the condition $\mathcal{C}(0) = \mathbf{0}$, then the filtration “by arity” (2.14) on $\text{Hom}(\mathcal{C}(A), B)$ satisfies the above conditions. In other words, $\text{Hom}(\mathcal{C}(A), B)$ is a filtered $\mathfrak{S}\text{Lie}_\infty$ -algebra.

The filtration (3.1) induces a natural descending filtration and hence a topology on $\underline{S}(L)$. Just as in [4], we tacitly assume that ∞ -morphisms of filtered $\mathfrak{S}\text{Lie}_\infty$ -algebras are continuous with respect to this topology.

Let us also recall [4] that an *enhanced morphism*

$$L_1 \xrightarrow{(\alpha, F)} L_2$$

between filtered $\mathfrak{S}\text{Lie}_\infty$ -algebras is a pair consisting of a MC element $\alpha \in L_2$ and a (continuous) ∞ -morphism $F : L_1 \rightarrow L_2^\alpha$, where L_2^α is obtained from L_2 via twisting⁵ by the MC element α . The composition of two enhanced morphisms $L_1 \xrightarrow{(\alpha_2, F)} L_2$ and $L_2 \xrightarrow{(\alpha_3, G)} L_3$ is the pair

$$(\alpha_3 + G_*(\alpha_2), G^{\alpha_2} \circ F), \quad (3.3)$$

where the ∞ -morphism G^{α_2} is obtained from G via twisting by the MC element $\alpha_2 \in L_2$.

Following [4], we denote by $\mathfrak{S}\text{Lie}_\infty^{\text{MC}}$ the category of filtered $\mathfrak{S}\text{Lie}_\infty$ -algebras with the above enhanced morphisms.

Given two filtered $\mathfrak{S}\text{Lie}_\infty$ algebras $(L, \{\cdot, \cdot\}, \{\cdot, \cdot, \cdot\}, \dots)$ and $(\tilde{L}, \{\cdot, \cdot\}^\sim, \{\cdot, \cdot, \cdot\}^\sim, \dots)$, one obtains a filtered $\mathfrak{S}\text{Lie}_\infty$ structure on the direct sum $L \oplus \tilde{L}$ by setting

$$\{x_1 + x'_1, x_2 + x'_2, \dots, x_k + x'_k\} := \{x_1, x_2, \dots, x_k\} + \{x'_1, x'_2, \dots, x'_k\}^\sim,$$

and

$$\mathcal{F}_k(L \oplus \tilde{L}) := (\mathcal{F}_k L) \oplus (\mathcal{F}_k \tilde{L}).$$

If α and $\tilde{\alpha}$ are MC elements of L and \tilde{L} , respectively, then $\alpha + \tilde{\alpha} \in L \oplus \tilde{L}$ is clearly a MC element of the $\mathfrak{S}\text{Lie}_\infty$ -algebra $L \oplus \tilde{L}$. Furthermore, the operation of twisting (by a MC element) is compatible with \oplus , i.e. the $\mathfrak{S}\text{Lie}_\infty$ -algebra $L^\alpha \oplus \tilde{L}^{\tilde{\alpha}}$ is canonically isomorphic to the $\mathfrak{S}\text{Lie}_\infty$ -algebra $(L \oplus \tilde{L})^{\alpha + \tilde{\alpha}}$.

Using these observations, we show, in [4, Section 3.1], that the assignment

$$(L, \tilde{L}) \mapsto L \oplus \tilde{L}$$

can be upgraded to a structure of a symmetric monoidal category on $\mathfrak{S}\text{Lie}_\infty^{\text{MC}}$ with $\mathbf{0}$ being the unit object.

3.2 The $\mathfrak{S}\text{Lie}_\infty^{\text{MC}}$ -enriched category of $\text{Cobar}(\mathcal{C})$ -algebras

Given two $\text{Cobar}(\mathcal{C})$ -algebras A and B , we denote by

$$\mathbf{map}(A, B) := \text{Hom}(\mathcal{C}(A), B) \quad (3.4)$$

the convolution $\mathfrak{S}\text{Lie}_\infty$ -algebra defined in Proposition 2.6.

In this section, we construct a $\mathfrak{S}\text{Lie}_\infty^{\text{MC}}$ -enriched category [4], [17] $\text{HoAlg}_\mathcal{C}$ whose objects are homotopy algebras A, B, \dots of type \mathcal{C} and whose mapping spaces are $\mathfrak{S}\text{Lie}_\infty$ -algebras (3.4).

⁵See [4, Section 2] for details on twisting by MC elements.

We start with defining a degree 0 linear map

$$\mathcal{U}' : \underline{S}(\text{Hom}(\mathcal{C}(A_2), A_3) \oplus \text{Hom}(\mathcal{C}(A_1), A_2)) \rightarrow \text{Hom}(\mathcal{C}(A_1), A_3) \quad (3.5)$$

by using the identification

$$\begin{aligned} \underline{S}(\text{Hom}(\mathcal{C}(A_2), A_3) \oplus \text{Hom}(\mathcal{C}(A_1), A_2)) &= \underline{S}(\text{Hom}(\mathcal{C}(A_2), A_3)) \oplus \underline{S}(\text{Hom}(\mathcal{C}(A_1), A_2)) \\ &\quad \oplus \left(\underline{S}(\text{Hom}(\mathcal{C}(A_2), A_3)) \otimes \underline{S}(\text{Hom}(\mathcal{C}(A_1), A_2)) \right) \end{aligned}$$

and the formulas:

$$\mathcal{U}'(g \otimes (f_1 \dots f_n))(X) = g(1 \otimes f_1 \otimes \dots \otimes f_n(\Delta_n(X))), \quad (3.6)$$

for all $X \in \mathcal{C}(A_1)$, $g \in \text{Hom}(\mathcal{C}(A_2), A_3)$, $f_1, \dots, f_n \in \text{Hom}(\mathcal{C}(A_1), A_2)$, and

$$\begin{aligned} \mathcal{U}'|_{\underline{S}(\text{Hom}(\mathcal{C}(A_2), A_3))} &= \mathcal{U}'|_{\underline{S}(\text{Hom}(\mathcal{C}(A_1), A_2))} = 0, \\ \mathcal{U}'|_{\underline{S}^{m \neq 1}(\text{Hom}(\mathcal{C}(A_2), A_3)) \otimes \underline{S}(\text{Hom}(\mathcal{C}(A_1), A_2))} &= 0. \end{aligned} \quad (3.7)$$

We claim that

Proposition 3.1 *The vector \mathcal{U}' defined by the above formulas is a MC element of the \mathfrak{SLie}_∞ -algebra*

$$\text{Hom}\left(\underline{S}(\text{Hom}(\mathcal{C}(A_2), A_3) \oplus \text{Hom}(\mathcal{C}(A_1), A_2)), \text{Hom}(\mathcal{C}(A_1), A_3)\right). \quad (3.8)$$

Proof. Let us denote by L_{ij} the \mathfrak{SLie}_∞ -algebra $\text{Hom}(\mathcal{C}(A_i), A_j)$ and by d_{ij} the differential on L_{ij} . We also denote by Q_{ij} the degree 1 coderivation on the \mathbf{coCom} -coalgebra

$$\underline{S}(L_{ij}) \quad (3.9)$$

corresponding to the \mathfrak{SLie}_∞ -algebra L_{ij} . By abuse of notation, d_{ij} also denotes the differential on (3.9) coming from the one on L_{ij} .

In terms of this notation, our goal is to prove that \mathcal{U}' satisfies the equation⁶

$$d_{13} \circ \mathcal{U}' - \mathcal{U}' \circ ((d_{23} + Q_{23}) \otimes 1 + 1 \otimes (d_{12} + Q_{12})) + \sum_{m=2}^{\infty} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m = 0. \quad (3.10)$$

We will present the most bulky part of the proof of (3.10). Namely, we will show in detail that the sum

$$\sum_{m=2}^{\infty} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m \quad (3.11)$$

cancels with the term $-\mathcal{U}' \circ (Q_{23} \otimes 1)$ in (3.10). The remaining cancellations are much more straightforward and we leave them to the reader.

⁶We sometimes use the subscript m in $\{ , \dots, \}_m$ to denote the number of entries of the corresponding multi-bracket.

In the computations given below, we do not specify explicitly sign factors coming from the Koszul sign rule. We address this issue by a short comment at the end of the proof.

Let $g_1, \dots, g_k \in L_{23}$ and $f_1, \dots, f_n \in L_{12}$. Due to (3.7),

$$\frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_k, f_1, \dots, f_n) = 0$$

if $k \neq m$ or $n < k$.

Unfolding the term

$$\frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n) \quad (3.12)$$

with $n \geq m$, we get

$$\begin{aligned} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n) = \\ \sum_{\substack{k_1 + \dots + k_m = n \\ k_j \geq 1}} \sum_{\tau \in S_m} \sum_{\sigma \in \text{Sh}_{k_1, k_2, \dots, k_m}} \frac{\pm 1}{m!} \{\mathcal{U}'(g_{\tau(1)}, f_{\sigma(1)}, \dots, f_{\sigma(k_1)}), \mathcal{U}'(g_{\tau(2)}, f_{\sigma(k_1+1)}, \dots, f_{\sigma(k_1+k_2)}), \dots \\ \dots \mathcal{U}'(g_{\tau(m)}, f_{\sigma(n-k_m+1)}, \dots, f_{\sigma(n)})\}_m^{L_{13}}, \end{aligned} \quad (3.13)$$

where $\{\ , \dots, \}_m^{L_{13}}$ denotes the corresponding multi-bracket on L_{13} and the sign factors in the right hand side are determined by the Koszul rule.

Since $\{\ , \dots, \}_m^{L_{13}}$ is (graded) symmetric in its argument, we can simplify term (3.12) further:

$$\begin{aligned} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n) = \\ \sum_{\substack{k_1 + \dots + k_m = n \\ k_j \geq 1}} \sum_{\sigma \in \text{Sh}_{k_1, k_2, \dots, k_m}} \pm \{\mathcal{U}'(g_1, f_{\sigma(1)}, \dots, f_{\sigma(k_1)}), \mathcal{U}'(g_2, f_{\sigma(k_1+1)}, \dots, f_{\sigma(k_1+k_2)}), \dots \\ \dots \mathcal{U}'(g_m, f_{\sigma(n-k_m+1)}, \dots, f_{\sigma(n)})\}_m^{L_{13}}. \end{aligned} \quad (3.14)$$

Let $X \in \mathcal{C}(A_1)$ and

$$\Delta_m(X) = \sum_{\alpha} (\gamma_{\alpha}; X_{\alpha,1}, X_{\alpha,2}, \dots, X_{\alpha,m}) \in (\mathcal{C}(m) \otimes \mathcal{C}(A_1)^{\otimes m})^{S_m}. \quad (3.15)$$

Then, applying equation (3.14) and using the definition of the multi-bracket on $L_{13} = \text{Hom}(\mathcal{C}(A_1), A_3)$, we get

$$\begin{aligned} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n)(X) = \\ \sum_{\alpha} \sum_{\substack{k_1 + \dots + k_m = n \\ k_j \geq 1}} \sum_{\sigma \in \text{Sh}_{k_1, k_2, \dots, k_m}} \pm Q_{A_3}(\gamma_{\alpha}; \mathcal{U}'(g_1, f_{\sigma(1)}, \dots, f_{\sigma(k_1)})(X_{\alpha,1}), \\ \mathcal{U}'(g_2, f_{\sigma(k_1+1)}, \dots, f_{\sigma(k_1+k_2)})(X_{\alpha,2}), \dots, \mathcal{U}'(g_m, f_{\sigma(n-k_m+1)}, \dots, f_{\sigma(n)})(X_{\alpha,m})) , \end{aligned} \quad (3.16)$$

where Q_{A_3} denotes the coderivation of $\mathcal{C}(A_3)$ corresponding to the $\text{Cobar}(\mathcal{C})$ -algebra structure on A_3 .

Using (3.6), we deduce that

$$\begin{aligned} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n)(X) = \\ \sum_{\substack{k_1 + \dots + k_m = n \\ k_j \geq 1}} \sum_{\sigma \in \text{Sh}_{k_1, k_2, \dots, k_m}} \pm Q_{A_3} \circ (1 \otimes g_1 \otimes f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(k_1)} \otimes \dots \otimes g_m \otimes f_{\sigma(n-k_m+1)} \otimes \dots \otimes f_{\sigma(n)}) \\ \circ (1 \otimes \Delta_{k_1} \otimes \Delta_{k_2} \otimes \dots \otimes \Delta_{k_m}) \circ \Delta_m(X). \end{aligned} \quad (3.17)$$

By the axioms of the \mathcal{C} -coalgebra structure on $\mathcal{C}(A_1)$, we have

$$(1 \otimes \Delta_{k_1} \otimes \Delta_{k_2} \otimes \dots \otimes \Delta_{k_m}) \circ \Delta_m(X) = \mathfrak{b} \circ (\Delta_{\mathbf{t}_{k_1, \dots, k_m}^\natural} \otimes 1^{\otimes n}) \circ \Delta_n(X), \quad (3.18)$$

where $n = k_1 + k_2 + \dots + k_m$, $\Delta_{\mathbf{t}_{k_1, \dots, k_m}^\natural}$ is the cooperadic comultiplication

$$\Delta_{\mathbf{t}_{k_1, \dots, k_m}^\natural} : \mathcal{C}(n) \rightarrow \mathcal{C}(m) \otimes \mathcal{C}(k_1) \otimes \mathcal{C}(k_2) \otimes \dots \otimes \mathcal{C}(k_m)$$

corresponding to the planar tree $\mathbf{t}_{k_1, \dots, k_m}^\natural$ shown on figure 2, and \mathfrak{b} is the braiding isomorphism which “changes the positions” of tensor factors appropriately.

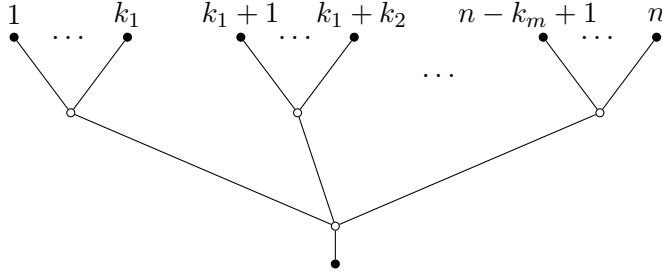


Figure 2: The labeled planar tree $\mathbf{t}_{k_1, \dots, k_m}^\natural$

Unfolding $\Delta_n(X)$

$$\Delta_n(X) = \sum_{\beta} (\tilde{\gamma}_{\beta}; \tilde{X}_{\beta,1}, \dots, \tilde{X}_{\beta,n})$$

and using the fact that Δ_n lands in the space of S_n -invariants, we rewrite the expression $(\Delta_{\mathbf{t}_{k_1, \dots, k_m}^\natural} \otimes 1^{\otimes n}) \circ \Delta_n(X)$ as follows:

$$\begin{aligned} (\Delta_{\mathbf{t}_{k_1, \dots, k_m}^\natural} \otimes 1^{\otimes n}) \circ \Delta_n(X) = \\ \sum_{\beta} \pm (\Delta_{\mathbf{t}_{k_1, \dots, k_m}^\natural} \otimes 1^{\otimes n})(\sigma^{-1}(\tilde{\gamma}_{\beta}); \tilde{X}_{\beta, \sigma(1)}, \dots, \tilde{X}_{\beta, \sigma(n)}) = \\ \sum_{\beta} \pm (\Delta_{\sigma(\mathbf{t}_{k_1, \dots, k_m}^\natural)} \otimes 1^{\otimes n})(\tilde{\gamma}_{\beta}; \tilde{X}_{\beta, \sigma(1)}, \dots, \tilde{X}_{\beta, \sigma(n)}) \end{aligned} \quad (3.19)$$

for any $\sigma \in S_n$.

Combining this observation with (3.18), we conclude that

$$\begin{aligned} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n)(X) \\ = \sum_{\beta} \sum_{\substack{k_1 + \dots + k_m = n \\ k_j \geq 1}} \sum_{\sigma \in \text{Sh}_{k_1, k_2, \dots, k_m}} \pm Q_{A_3} \circ (1 \otimes g_1 \otimes \dots \otimes g_m) \\ \circ (\Delta_{\sigma(\mathbf{t}_{k_1, \dots, k_m}^{\text{fl}})} \otimes 1^{\otimes n})(\tilde{\gamma}_{\beta}; f_{\sigma(1)}(\tilde{X}_{\beta, \sigma(1)}), \dots, f_{\sigma(n)}(\tilde{X}_{\beta, \sigma(n)})) . \end{aligned} \quad (3.20)$$

On the other hand,

$$\begin{aligned} \sum_{\beta} \sum_{\substack{k_1 + \dots + k_m = n \\ k_j \geq 1}} \sum_{\sigma \in \text{Sh}_{k_1, k_2, \dots, k_m}} \pm (\Delta_{\sigma(\mathbf{t}_{k_1, \dots, k_m}^{\text{fl}})} \otimes 1^{\otimes n})(\tilde{\gamma}_{\beta}; f_{\sigma(1)}(\tilde{X}_{\beta, \sigma(1)}), \dots, f_{\sigma(n)}(\tilde{X}_{\beta, \sigma(n)})) \\ = \Delta_m \left(\sum_{\beta} \pm (\tilde{\gamma}_{\beta}; f_1(\tilde{X}_{\beta, 1}), \dots, f_n(\tilde{X}_{\beta, n})) \right) . \end{aligned} \quad (3.21)$$

Therefore, by definition of the \mathfrak{SLie}_{∞} -structure on $L_{23} = \text{Hom}(\mathcal{C}(A_2), A_3)$, we have

$$\begin{aligned} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n)(X) = \\ \mathcal{U}'(\{g_1, \dots, g_m\}_m^{L_{23}}, f_1, \dots, f_n)(X) . \end{aligned} \quad (3.22)$$

Let us also observe that, due to (3.7),

$$(\mathcal{U}' \circ (Q_{23} \otimes 1))(g_1, \dots, g_m, f_1, \dots, f_n)(X) = \mathcal{U}'(\{g_1, \dots, g_m\}_m^{L_{23}}, f_1, \dots, f_n)(X) .$$

Thus,

$$\begin{aligned} \frac{1}{m!} \{\mathcal{U}', \mathcal{U}', \dots, \mathcal{U}'\}_m(g_1, \dots, g_m, f_1, \dots, f_n)(X) = \\ (\mathcal{U}' \circ (Q_{23} \otimes 1))(g_1, \dots, g_m, f_1, \dots, f_n)(X) \end{aligned} \quad (3.23)$$

which implies the desired cancellation of sum (3.11) with the term $-\mathcal{U}' \circ (Q_{23} \otimes 1)$ in (3.10).

Let us now address the issue of sign factors. The sign factor in front of the term

$$Q_{A_3} \circ (1 \otimes g_1 \otimes \dots \otimes g_m) \circ (\Delta_{\sigma(\mathbf{t}_{k_1, \dots, k_m}^{\text{fl}})} \otimes 1^{\otimes n})(\tilde{\gamma}_{\beta}; f_{\sigma(1)}(\tilde{X}_{\beta, \sigma(1)}), \dots, f_{\sigma(n)}(\tilde{X}_{\beta, \sigma(n)})) \quad (3.24)$$

in (3.20) comes from rearranging the homogeneous vectors

$$f_1, f_2, \dots, f_n, \tilde{\gamma}_{\beta}, \tilde{X}_{\beta, 1}, \tilde{X}_{\beta, 2}, \dots, \tilde{X}_{\beta, n} \quad (3.25)$$

from their standard order in (3.25) to the order in which they appear in (3.24). It is easy to see that we get the same sign factors in front of the corresponding terms, when we unfold the right hand side of (3.22).

Proposition 3.1 is proved. \square

Combining Lemma 2.9 with Proposition 3.1 we deduce that

Corollary 3.2 *The map \mathcal{U}' defined by equations (3.6) and (3.7) lifts to an ∞ -morphism*

$$\mathcal{U} : \mathbf{map}(A_2, A_3) \oplus \mathbf{map}(A_1, A_2) \rightarrow \mathbf{map}(A_1, A_3). \quad (3.26)$$

□

Remark 3.3 We would like to remark that the map

$$\mathcal{U}' : \underline{S}(\mathrm{Hom}(\mathcal{C}(A_2), A_3) \oplus \mathrm{Hom}(\mathcal{C}(A_1), A_2)) \rightarrow \mathrm{Hom}(\mathcal{C}(A_1), A_3),$$

can be equivalently defined by the single formula:

$$\mathcal{U}'((g_1 \oplus f_1), (g_2 \oplus f_2), \dots, (g_n \oplus f_n)) = \sum_{i=1}^n \pm g_i \circ (1 \otimes f_1 \otimes \dots \otimes \widehat{f_i} \otimes \dots \otimes f_n) \circ \Delta_{n-1} \quad (3.27)$$

where $(g_i \oplus f_i) \in \mathrm{Hom}(\mathcal{C}(A_2), A_3) \oplus \mathrm{Hom}(\mathcal{C}(A_1), A_2)$ and \pm is the usual Koszul sign factor.

The following theorem shows that the composition in $\mathbf{HoAlg}_{\mathcal{C}}$ given by ∞ -morphism (3.26) is associative.

Theorem 3.4 *Let A_1, \dots, A_4 be $\mathrm{Cobar}(\mathcal{C})$ -algebras and let*

$$\mathcal{U}_{i_1 i_2 i_3} : \mathbf{map}(A_{i_2}, A_{i_3}) \oplus \mathbf{map}(A_{i_1}, A_{i_2}) \rightarrow \mathbf{map}(A_{i_1}, A_{i_3})$$

be the ∞ -morphism given in Corollary 3.2. Then the following diagram

$$\begin{array}{ccc} & \mathbf{map}(A_3, A_4) \oplus \mathbf{map}(A_1, A_3) & \\ \mathrm{id} \otimes \mathcal{U}_{123} \nearrow & & \searrow \mathcal{U}_{134} \\ \mathbf{map}(A_3, A_4) \oplus (\mathbf{map}(A_2, A_3) \oplus \mathbf{map}(A_1, A_2)) & & \mathbf{map}(A_1, A_4) \\ \downarrow \cong & & \nearrow \mathcal{U}_{124} \\ (\mathbf{map}(A_3, A_4) \oplus \mathbf{map}(A_2, A_3)) \oplus \mathbf{map}(A_1, A_2) & & \\ \searrow \mathcal{U}_{234} \otimes \mathrm{id} & \mathbf{map}(A_2, A_4) \oplus \mathbf{map}(A_1, A_2) & \end{array} \quad (3.28)$$

commutes.

Proof. Let $h \in \mathbf{map}(A_3, A_4)$, $g_1, \dots, g_m \in \mathbf{map}(A_2, A_3)$, and $f_1, \dots, f_n \in \mathbf{map}(A_1, A_2)$. Composing the lower arrows in (3.28) with the canonical projection

$$p_{14} : \underline{S}(\mathbf{map}(A_1, A_4)) \rightarrow \mathbf{map}(A_1, A_4), \quad (3.29)$$

we get

$$\begin{aligned} p_{14} \circ \mathcal{U}_{124} \circ (\mathcal{U}_{234} \otimes \mathrm{id})(h, g_1, \dots, g_m, f_1, \dots, f_n) = \\ h \left((1 \otimes g_1 \otimes \dots \otimes g_m) \circ \Delta_m \circ (1 \otimes f_1 \otimes \dots \otimes f_n) \circ \Delta_n \right). \end{aligned} \quad (3.30)$$

Similarly, composing the upper arrows in (3.28) with canonical projection (3.29), we get

$$\begin{aligned}
p_{14} \circ \mathcal{U}_{134} \circ (\text{id} \otimes \mathcal{U}_{123})(h, g_1, \dots, g_m, f_1, \dots, f_n) = \\
\sum_{\substack{k_1 + \dots + k_m = n \\ \sigma \in \text{Sh}(k_1, \dots, k_m)}} \pm h \left(1 \otimes g_1 (1 \otimes f_{\sigma(1)} \otimes f_{\sigma(2)} \otimes \dots \otimes f_{\sigma(k_1)}) \otimes g_2 (1 \otimes f_{\sigma(k_1+1)} \otimes \dots \otimes f_{\sigma(k_1+k_2)}) \otimes \dots \right. \\
\left. \otimes g_m (1 \otimes f_{\sigma(n-k_m+1)} \otimes \dots \otimes f_{\sigma(n)}) \circ (1 \otimes \Delta_{k_1} \otimes \Delta_{k_2} \otimes \dots \otimes \Delta_{k_m}) \right) \Delta_m,
\end{aligned} \tag{3.31}$$

where the sign factors are determined by the Koszul rule.

Using equation (3.18), computation (3.19), and equation (3.21) from the proof of Proposition 3.1, we conclude that the left hand side of (3.30) coincides with the left hand side of (3.31). Since any ∞ -morphism to $\mathbf{map}(A_1, A_4)$ is uniquely determined by its composition with projection (3.29), we deduce that

$$\mathcal{U}_{124} \circ (\mathcal{U}_{234} \otimes \text{id}) = \mathcal{U}_{134} \circ (\text{id} \otimes \mathcal{U}_{123}).$$

Thus diagram (3.28) is indeed commutative. \square

3.3 The proof of the unit axiom

Let us recall that $\mathbf{0}$ is the unit object in the category $\mathfrak{SLie}_\infty^{\text{MC}}$ and observe that for every Cobar(\mathcal{C})-algebra A we have a canonical enhanced morphism

$$\mathbf{0} \xrightarrow{(\text{id}_A, 0)} \mathbf{map}(A, A), \tag{3.32}$$

where id_A is the MC element of $\mathbf{map}(A, A) = \text{Hom}(\mathcal{C}(A), A)$ corresponding to the identity ∞ -morphism

$$\text{id}_{\mathcal{C}(A)} : \mathcal{C}(A) \rightarrow \mathcal{C}(A)$$

and 0 is the unique \mathfrak{SLie}_∞ -morphism from $\mathbf{0}$ to $\mathbf{map}(A, A)$.

We claim that

Proposition 3.5 *For every pair A, B of homotopy algebras of type \mathcal{C} , the diagrams*

$$\begin{array}{ccc}
\mathbf{map}(A, B) \oplus \mathbf{map}(A, A) & \xrightarrow{\mathcal{U}} & \mathbf{map}(A, B) \\
\uparrow \text{id}_{\mathbf{map}(A, B)} \otimes (\text{id}_A, 0) & \nearrow & \\
\mathbf{map}(A, B) \oplus \mathbf{0} & &
\end{array} \tag{3.33}$$

$$\begin{array}{ccc}
\mathbf{map}(B, B) \oplus \mathbf{map}(A, B) & \xrightarrow{\mathcal{U}} & \mathbf{map}(A, B) \\
\uparrow (\text{id}_B, 0) \otimes \text{id}_{\mathbf{map}(A, B)} & \nearrow & \\
\mathbf{0} \oplus \mathbf{map}(A, B) & &
\end{array} \tag{3.34}$$

commute.

Proof. Let us denote by

$$\mathcal{K} : \mathbf{map}(A, B) \oplus \mathbf{0} \rightarrow \mathbf{map}(A, B)$$

the composition of the vertical arrow and the horizontal arrow in (3.33) and let \mathcal{K}' be the corresponding element in

$$\mathrm{Hom}(\underline{\mathcal{S}}(\mathbf{map}(A, B) \oplus \mathbf{0}), \mathbf{map}(A, B)).$$

Unfolding the definition of composition of enhanced morphisms in $\mathfrak{SLie}_\infty^{\mathrm{MC}}$ (see Proposition 3.4 in [4]), we get that

$$\mathcal{K}'(g_1, \dots, g_m) = \sum_{n \geq 0} \frac{1}{n!} \mathcal{U}'((g_1, \dots, g_m) \otimes \mathrm{id}_A^n), \quad (3.35)$$

where $g_1, \dots, g_m \in \mathbf{map}(A, B)$.

Hence, using (3.6) and (3.7) we deduce that

$$\mathcal{K}'(g_1, \dots, g_m) = 0 \quad \forall \quad m \geq 2$$

and, for every $X \in \mathcal{C}(A)$,

$$\mathcal{K}'(g_1)(X) = \sum_{n \geq 0} \frac{1}{n!} g_1((1 \otimes \mathrm{id}_A^n) \Delta_n(X)) = g_1(X), \quad (3.36)$$

where the last equality follows from the identification of $\mathcal{C}(\mathcal{C}(A))^{\mathrm{inv}}$ and $\mathcal{C}(\mathcal{C}(A))$ via the inverse of isomorphism (1.2).

Thus diagram (3.33) indeed commutes.

The proof of the commutativity of (3.34) is easier. So we leave it to the reader. \square

3.4 The simplicial category $\mathrm{HoAlg}_\mathcal{C}^\Delta$ of homotopy algebras

Let $\Omega_n = \Omega^\bullet(\Delta^n)$ denote the polynomial de Rham complex on the n -simplex with coefficients in \mathbb{k} , and $\{\Omega_n\}_{n \geq 0}$ the associated simplicial dg commutative \mathbb{k} -algebra. Let us recall [4, Proposition 4.1] that for every filtered \mathfrak{SLie}_∞ -algebra L the simplicial set $\mathfrak{MC}_\bullet(L)$ with⁷

$$\mathfrak{MC}_n(L) := \mathrm{MC}(L \hat{\otimes} \Omega_n)$$

is a Kan complex (a.k.a. an ∞ -groupoid). We call the simplicial set $\mathfrak{MC}_\bullet(L)$ the *Deligne-Getzler-Hinich (DGH) ∞ -groupoid*.

Let us also recall (see Theorem 4.2 in [4]) that applying the functor \mathfrak{MC}_\bullet to mapping spaces of any $\mathfrak{SLie}_\infty^{\mathrm{MC}}$ -enriched category, we get a simplicial category whose mapping spaces are Kan complexes. Thus, applying [4, Theorem 4.2] to the $\mathfrak{SLie}_\infty^{\mathrm{MC}}$ -enriched category $\mathrm{HoAlg}_\mathcal{C}$ and using Lemma 2.9 we deduce the following theorem:

⁷Recall that $\mathrm{MC}(L)$ denotes the set of MC elements of a filtered \mathfrak{SLie}_∞ -algebras L [4].

Theorem 3.6 *Let \mathcal{C} be a coaugmented dg cooperad satisfying Conditions (2.1) and (2.15). Then the assignment*

$$(A, B) \in \text{Objects}(\text{Cat}_{\mathcal{C}}) \times \text{Objects}(\text{Cat}_{\mathcal{C}}) \mapsto \mathfrak{MC}_{\bullet}(\mathbf{map}(A, B))$$

gives us a category enriched over ∞ -groupoids (a.k.a. Kan complexes). Moreover, for every pair of $\text{Cobar}(\mathcal{C})$ -algebras A, B , the set $\mathfrak{MC}_0(\mathbf{map}(A, B))$ is in bijection with the set of ∞ -morphisms from A to B . \square

4 $\pi_0(\text{HoAlg}_{\mathcal{C}}^{\Delta})$ is a correct homotopy category of homotopy algebras

Let A and B be homotopy algebras of type \mathcal{C} and F be an ∞ -morphism from A to B :

$$F : (\mathcal{C}(A), \partial + Q_A) \rightarrow (\mathcal{C}(B), \partial + Q_B).$$

Composing F with a canonical projection $p_B : \mathcal{C}(B) \rightarrow B$, and restricting this composition to $A \subset \mathcal{C}(A)$, we get a map of cochain complexes:

$$p_B \circ F \Big|_A : A \rightarrow B. \quad (4.1)$$

We refer to (4.1) as *the linear term* of the ∞ -morphism F . Recall that an ∞ -morphism F is called an ∞ *quasi-isomorphism* if its linear term (4.1) is a quasi-isomorphism of cochain complexes.

Let us recall that $\text{Cat}_{\mathcal{C}}$ is the category of $\text{Cobar}(\mathcal{C})$ -algebras with morphisms being ∞ -morphisms, and observe that we have the obvious functor

$$\mathfrak{F} : \text{Cat}_{\mathcal{C}} \rightarrow \pi_0(\text{HoAlg}_{\mathcal{C}}^{\Delta}) \quad (4.2)$$

which acts by identity on objects and assigns to every ∞ -morphism the isomorphism class of the corresponding MC element.

The goal of this section is to prove that the category $\pi_0(\text{HoAlg}_{\mathcal{C}}^{\Delta})$ is the homotopy category for $\text{Cat}_{\mathcal{C}}$. Namely,

Theorem 4.1 *The functor \mathfrak{F} sends ∞ quasi-isomorphisms to isomorphisms and it is a universal functor with this property. I.e., if $\mathfrak{G} : \text{Cat}_{\mathcal{C}} \rightarrow \mathfrak{D}$ is a functor which sends ∞ quasi-isomorphisms to isomorphisms in \mathfrak{D} then there exists a unique functor*

$$\mathfrak{G}' : \pi_0(\text{HoAlg}_{\mathcal{C}}^{\Delta}) \rightarrow \mathfrak{D}$$

such that $\mathfrak{G} = \mathfrak{G}' \circ \mathfrak{F}$.

4.1 “Inverting” ∞ quasi-isomorphisms

Let A_1, A_2, A_3 be $\text{Cobar}(\mathcal{C})$ -algebras, F be an ∞ -morphism from A_1 to A_2 , and F' be the MC element of $\mathbf{map}(A_1, A_2)$ corresponding to F . Let us denote by $F' \oplus 0, 0 \oplus F'$ the corresponding MC elements of the \mathfrak{SLie}_∞ -algebras

$$\mathbf{map}(A_1, A_2) \oplus \mathbf{map}(A_3, A_1)$$

and

$$\mathbf{map}(A_2, A_3) \oplus \mathbf{map}(A_1, A_2),$$

respectively.

Twisting the \mathfrak{SLie}_∞ -morphisms

$$\mathcal{U} : \mathbf{map}(A_1, A_2) \oplus \mathbf{map}(A_3, A_1) \rightarrow \mathbf{map}(A_3, A_2)$$

and

$$\mathcal{U} : \mathbf{map}(A_2, A_3) \oplus \mathbf{map}(A_1, A_2) \rightarrow \mathbf{map}(A_1, A_3)$$

by the MC elements $F' \oplus 0, 0 \oplus F'$, respectively, and composing the resulting \mathfrak{SLie}_∞ -morphisms with the canonical maps

$$f \mapsto 0 \oplus f : \mathbf{map}(A_3, A_1) \rightarrow \mathbf{map}(A_1, A_2) \oplus \mathbf{map}(A_3, A_1)$$

$$f \mapsto f \oplus 0 : \mathbf{map}(A_2, A_3) \rightarrow \mathbf{map}(A_2, A_3) \oplus \mathbf{map}(A_1, A_2)$$

we get two \mathfrak{SLie}_∞ -morphisms

$$\mathcal{U}_{A_3 A_1 A_2} : \mathbf{map}(A_3, A_1) \rightarrow \mathbf{map}(A_3, A_2) \quad (4.3)$$

and

$$\mathcal{U}_{A_1 A_2 A_3} : \mathbf{map}(A_2, A_3) \rightarrow \mathbf{map}(A_1, A_3). \quad (4.4)$$

The following proposition says that the induced maps of MC elements

$$(\mathcal{U}_{A_3 A_1 A_2})_* : \text{MC}(\mathbf{map}(A_3, A_1)) \rightarrow \text{MC}(\mathbf{map}(A_3, A_2)) \quad (4.5)$$

and

$$(\mathcal{U}_{A_1 A_2 A_3})_* : \text{MC}(\mathbf{map}(A_2, A_3)) \rightarrow \text{MC}(\mathbf{map}(A_1, A_3)) \quad (4.6)$$

correspond to the composition (resp. the pre-composition) of an ∞ -morphism from A_3 to A_1 (resp. from A_2 to A_3) with F :

Proposition 4.2 *If G is an ∞ -morphism from A_3 to A_1 and G' is the corresponding MC element of $\mathbf{map}(A_3, A_1)$ then the MC element $(\mathcal{U}_{A_3 A_1 A_2})_*(G')$ of $\mathbf{map}(A_3, A_2)$ corresponds to the composition $F \circ G$. Similarly, if G is an ∞ -morphism from A_2 to A_3 and G' is the corresponding MC element of $\mathbf{map}(A_2, A_3)$ then the MC element $(\mathcal{U}_{A_1 A_2 A_3})_*(G')$ of $\mathbf{map}(A_1, A_3)$ corresponds to the composition $G \circ F$.*

Proof. The proof of these statements is straightforward. □

We also claim that

Proposition 4.3 $\mathfrak{S}\text{Lie}_\infty$ -morphisms (4.3) and (4.4) are compatible with the filtrations $\mathcal{F}_\bullet^{\text{ari}}$ from (2.14). Furthermore, if F is an ∞ quasi-isomorphism, then (4.3) and (4.4) give us $\mathfrak{S}\text{Lie}_\infty$ quasi-isomorphisms

$$\mathcal{F}_m^{\text{ari}} \mathbf{map}(A_3, A_1) \rightarrow \mathcal{F}_m^{\text{ari}} \mathbf{map}(A_3, A_2) \quad \text{and} \quad \mathcal{F}_m^{\text{ari}} \mathbf{map}(A_2, A_3) \rightarrow \mathcal{F}_m^{\text{ari}} \mathbf{map}(A_1, A_3) \quad (4.7)$$

respectively, for every $m \geq 1$.

Proof. Due to Remark 3.3, the composition

$$\mathcal{U}'_{A_3 A_1 A_2} := p_{\mathbf{map}(A_3, A_2)} \circ \mathcal{U}_{A_3 A_1 A_2} : \underline{\mathcal{S}}(\mathbf{map}(A_3, A_1)) \rightarrow \mathbf{map}(A_3, A_2)$$

is given by the formula

$$\mathcal{U}'_{A_3 A_1 A_2}(g_1, \dots, g_n)(X) = F'(1 \otimes g_1 \otimes \dots \otimes g_n(\Delta_n(X))), \quad (4.8)$$

where $X \in \mathcal{C}(A_3)$ and $g_1, \dots, g_n \in \mathbf{map}(A_3, A_1)$. Similarly, the composition

$$\mathcal{U}'_{A_1 A_2 A_3} := p_{\mathbf{map}(A_1, A_3)} \circ \mathcal{U}_{A_1 A_2 A_3} : \underline{\mathcal{S}}(\mathbf{map}(A_2, A_3)) \rightarrow \mathbf{map}(A_1, A_3)$$

is given by the formula

$$\mathcal{U}'_{A_1 A_2 A_3}(h_1, \dots, h_n)(Y) = \begin{cases} \sum_{m \geq 1} \frac{1}{m!} h_1(1 \otimes F' \otimes \dots \otimes F'(\Delta_m(Y))) & \text{if } n = 1, \\ 0 & \text{otherwise,} \end{cases} \quad (4.9)$$

where $Y \in \mathcal{C}(A_1)$ and $h_1, \dots, h_n \in \mathbf{map}(A_2, A_3)$.

The compatibility of $\mathbf{map} \mathcal{U}'_{A_3 A_1 A_2}$ and $\mathcal{U}'_{A_1 A_2 A_3}$ with the filtrations $\mathcal{F}_\bullet^{\text{ari}}$ can be checked directly by unfolding the right hand side of (4.8) and the right hand side of (4.9), respectively.

We also see that the linear terms

$$\mathcal{U}_{1, A_3 A_1 A_2} : \mathbf{map}(A_3, A_1) \rightarrow \mathbf{map}(A_3, A_2) \quad (4.10)$$

and

$$\mathcal{U}_{1, A_1 A_2 A_3} : \mathbf{map}(A_2, A_3) \rightarrow \mathbf{map}(A_1, A_3) \quad (4.11)$$

of the $\mathfrak{S}\text{Lie}_\infty$ -morphisms $\mathcal{U}_{A_3 A_1 A_2}$ and $\mathcal{U}_{A_1 A_2 A_3}$ are given by the formulas:

$$\mathcal{U}_{1, A_3 A_1 A_2}(g)(X) = F'((1 \otimes g) \circ \Delta_1(X))$$

and

$$\mathcal{U}_{1, A_1 A_2 A_3}(h)(Y) = \sum_{k \geq 1} \frac{1}{k!} h(1 \otimes F' \otimes \dots \otimes F'(\Delta_k(Y))),$$

respectively, where $g \in \mathbf{map}(A_3, A_1)$, $h \in \mathbf{map}(A_2, A_3)$, $X \in \mathcal{C}(A_3)$ and $Y \in \mathcal{C}(A_1)$.

Let $\varphi : A_1 \rightarrow A_2$ be the linear term

$$\varphi := p_{A_2} \circ F|_{A_1} \quad (4.12)$$

of the ∞ -morphism F and assume that φ is a quasi-isomorphism of cochain complexes.

To prove that $\mathcal{U}_{1,A_1A_2A_3}$ induces an isomorphism

$$H^\bullet(\mathcal{F}_m^{\text{ari}} \mathbf{map}(A_2, A_3)) \rightarrow H^\bullet(\mathcal{F}_m^{\text{ari}} \mathbf{map}(A_1, A_3))$$

for every m , we observe that, for $m \geq 2$

$$\mathcal{F}_m^{\text{ari}} \mathbf{map}(W, A_3) = \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3)$$

and $\mathbf{map}(W, A_3) = \mathcal{F}_1^{\text{ari}} \mathbf{map}(W, A_3)$ splits into the direct sum of cochain complexes

$$\mathbf{map}(W, A_3) = \text{Hom}(W, A_3) \oplus \text{Hom}(\mathcal{C}_\circ(W), A_3), \quad (4.13)$$

where W is either A_2 or A_1 , and \mathcal{C}_\circ is the cokernel of the coaugmentation of \mathcal{C} .

We also observe that the chain map $\mathcal{U}_{1,A_1A_2A_3}$ is compatible with decomposition (4.13) and the corresponding chain map

$$\text{Hom}(A_2, A_3) \rightarrow \text{Hom}(A_1, A_3)$$

is a quasi-isomorphism because the functor $\text{Hom}(-, -)$ preserves quasi-isomorphisms in $\mathbf{Ch}_\mathbb{k}$.

So we should now prove that the chain map

$$\mathcal{U}_{1,A_1A_2A_3} \Big|_{\mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(A_2), A_3)} : \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(A_2), A_3) \rightarrow \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(A_1), A_3) \quad (4.14)$$

induces an isomorphism on cohomology for every $m \geq 1$.

For this purpose, we equip the cochain complex $\mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3)$ with the following descending filtration

$$\begin{aligned} \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) &= \mathcal{F}_0^{\mathcal{C}} \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) \supset \mathcal{F}_1^{\mathcal{C}} \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) \supset \\ &\supset \mathcal{F}_2^{\mathcal{C}} \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) \supset \dots \\ \mathcal{F}_q^{\mathcal{C}} \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) &:= \\ \{g \in \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) \mid g(X) = 0 \ \forall X \in \mathcal{F}^q \mathcal{C}_\circ(W)\} &, \end{aligned} \quad (4.15)$$

where $\mathcal{F}^\bullet \mathcal{C}_\circ$ is the ascending filtration on the pseudo-cooperad \mathcal{C}_\circ from (2.1) and W is, as above, either A_1 or A_2 .

Due to inclusion (2.2), the differential on $\mathbf{map}(W, A_3)$ is compatible with the filtration. Moreover, condition (2.3) implies that $\mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3)$ is complete with respect to this filtration, i.e.

$$\mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) = \lim_q \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3) / \mathcal{F}_q^{\mathcal{C}} \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3).$$

Let us denote by $E_{m,q}(\mathcal{C}, W)$ the following cochain complex

$$E_{m,q}(\mathcal{C}_\circ, W) := \bigoplus_{n \geq m} \left((\mathcal{F}^q \mathcal{C}_\circ(n) / \mathcal{F}^{q-1} \mathcal{C}_\circ(n)) \otimes W^{\otimes n} \right)_{S_n}. \quad (4.16)$$

It is clear that, the associated graded complex

$$\text{Gr}_{\mathcal{F}^{\mathcal{C}}} \mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_\circ(W), A_3)$$

is isomorphic to

$$\bigoplus_{q \geq 1} \text{Hom}(E_{m,q}(\mathcal{C}_o, W), A_3). \quad (4.17)$$

Furthermore, the differential ∂_{Gr} on (4.17) comes from those on W , \mathcal{C}_o and A_3 .

The map between the associated graded complexes

$$\mathcal{U}_{1,A_1 A_2 A_3}^{\text{Gr}} : \text{Hom}(E_{m,q}(\mathcal{C}_o, A_2), A_3) \rightarrow \text{Hom}(E_{m,q}(\mathcal{C}_o, A_1), A_3) \quad (4.18)$$

induced by (4.14) is given by the formula

$$\mathcal{U}_{1,A_1 A_2 A_3}^{\text{Gr}}(g)(\gamma; a_1, \dots, a_n) = g(\gamma; \varphi(a_1), \dots, \varphi(a_n)), \quad (4.19)$$

where φ is the linear term of the ∞ -morphism F and $\gamma \in \mathcal{F}^q \mathcal{C}_o(n) / \mathcal{F}^{q-1} \mathcal{C}_o(n)$.

Using the Künneth theorem, the fact that φ induces an isomorphism $H^\bullet(A_1) \rightarrow H^\bullet(A_2)$, and $\text{char}(\mathbb{k}) = 0$ we conclude that the chain map

$$E_{m,q}(\mathcal{C}_o, A_1) \rightarrow E_{m,q}(\mathcal{C}_o, A_2)$$

induced by φ is a quasi-isomorphism.

Therefore, since the functor $\text{Hom}(-, -)$ preserves quasi-isomorphisms in $\mathbf{Ch}_{\mathbb{k}}$, we deduce chain map (4.18) between the associated graded complexes is a quasi-isomorphism.

Since $\mathcal{F}_m^{\text{ari}} \text{Hom}(\mathcal{C}_o(W), A_3)$ is complete with respect to filtration (4.15), and filtration (4.15) is bounded from the left, applying Lemma D.1 from [6] to the cone of chain map (4.14), we conclude that (4.14), and hence (4.11), is indeed a quasi-isomorphism of cochain complexes.

A similar argument shows that map (4.10) is a quasi-isomorphism of cochain complexes, provided so is φ (4.12).

Proposition 4.3 is proved. \square

Let A and B be $\text{Cobar}(\mathcal{C})$ -algebras. We will now prove that every ∞ quasi-morphism F from A to B is “invertible” in the following sense:

Corollary 4.4 *Let F be an ∞ quasi-isomorphism from A to B . Then there exists an ∞ -morphism G from B to A such that the MC element $(G \circ F)'$ (resp. $(F \circ G)'$) of the $\mathfrak{S}\text{Lie}_\infty$ -algebra $\mathbf{map}(A, A)$ (resp. $\mathbf{map}(B, B)$) corresponding to the composition $G \circ F$ (resp. $F \circ G$) is isomorphic in $\mathfrak{MC}_\bullet(\mathbf{map}(A, A))$ (resp. in $\mathfrak{MC}_\bullet(\mathbf{map}(B, B))$) to the MC element id'_A (resp. id'_B) corresponding to the identity morphism id_A (resp. id_B). If \tilde{G} is another ∞ -morphism from B to A satisfying the above properties then the MC element $\tilde{G}' \in \mathbf{map}(B, A)$ corresponding to \tilde{G} is isomorphic to G' in $\mathfrak{MC}_\bullet(\mathbf{map}(B, A))$.*

Proof. Let us start with the question of existence of G .

Due to Proposition 4.2, it suffices to prove that there exists a MC element G' of $\mathbf{map}(B, A)$ such that the 0-cell

$$(\mathcal{U}_{ABA})_*(G') \in \mathfrak{MC}_0(\mathbf{map}(A, A)) \quad (4.20)$$

is connected to id'_A and the 0-cell

$$(\mathcal{U}_{BAB})_*(G') \in \mathfrak{MC}_0(\mathbf{map}(B, B)) \quad (4.21)$$

is connected to id'_B .

Proposition 4.3 implies that the \mathfrak{SLie}_∞ -morphism

$$\mathcal{U}_{ABA} : \mathbf{map}(B, A) \rightarrow \mathbf{map}(A, A)$$

is a quasi-isomorphism and, moreover, it satisfies the necessary conditions of Theorem 2.2 from [2]. This theorem, in turn, implies that \mathcal{U}_{ABA} induces a bijection of sets

$$\pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(B, A))\right) \xrightarrow{\cong} \pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(A, A))\right). \quad (4.22)$$

Therefore, there exists a MC element $G' \in \mathbf{map}(B, A)$ such that the 0-cell

$$(G \circ F)' = (\mathcal{U}_{ABA})_*(G') \quad (4.23)$$

is connected to id'_A .

To prove that the 0-cell $(F \circ G)' = (\mathcal{U}_{BAB})_*(G')$ is connected to id'_B , we consider the \mathfrak{SLie}_∞ -morphisms

$$\mathcal{U}_{ABB} : \mathbf{map}(B, B) \rightarrow \mathbf{map}(A, B), \quad (4.24)$$

and

$$\mathcal{U}_{AAB} : \mathbf{map}(A, A) \rightarrow \mathbf{map}(A, B) \quad (4.25)$$

constructed, as above, using the ∞ -morphism F .

Proposition 4.2 implies that, if K' is a MC element of $\mathbf{map}(B, B)$ (resp. $\mathbf{map}(A, A)$) corresponding to an ∞ -morphism K from B to B (resp. from A to A) then the MC element $(\mathcal{U}_{ABB})_*(K')$ (resp. $(\mathcal{U}_{AAB})_*(K')$) corresponds to the composition $K \circ F$ (resp. $F \circ K$).

Let us now consider the composition $F \circ G$ and denote by $(F \circ G)'$ the 0-cell of $\mathfrak{MC}_\bullet(\mathbf{map}(B, B))$ corresponding to the ∞ -morphism $F \circ G$.

Due to the above observation, the MC element $(F \circ G \circ F)' \in \mathbf{map}(A, B)$ corresponding to the ∞ -morphism $F \circ G \circ F$ satisfies the equations

$$(F \circ G \circ F)' = (\mathcal{U}_{ABB})_*(F \circ G)' \quad (4.26)$$

and

$$(F \circ G \circ F)' = (\mathcal{U}_{AAB})_*(G \circ F)'. \quad (4.27)$$

Therefore, since the 0-cell $(G \circ F)'$ is connected to the 0-cell id'_A , the 0-cell $(F \circ G \circ F)'$ is connected to F' in $\mathfrak{MC}_\bullet(\mathbf{map}(A, B))$.

On other hand, $F' = (\mathcal{U}_{ABB})_*(\text{id}'_B)$, and hence the 0-cells

$$(\mathcal{U}_{ABB})_*(\text{id}'_B) \quad \text{and} \quad (\mathcal{U}_{ABB})_*(F \circ G)'$$

are connected in $\mathfrak{MC}_\bullet(\mathbf{map}(A, B))$.

Since F is an ∞ quasi-isomorphism, Proposition 4.3 and [2, Theorem 2.2] imply that $(\mathcal{U}_{ABB})_*$ induces a bijection

$$\pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(B, B))\right) \xrightarrow{\cong} \pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(A, B))\right).$$

Thus the 0-cells $(F \circ G)'$ and id'_B are also connected in $\mathfrak{MC}_\bullet(\mathbf{map}(B, B))$ and the existence of a desired ∞ -morphism G is proved.

Let \tilde{G} be another ∞ -morphism from B to A such that 0-cells (4.23) and id'_A are connected in $\mathfrak{MC}_\bullet(\mathbf{map}(A, A))$.

Therefore, since \mathcal{U}_{ABA} induces bijection (4.22) and the 0-cells $(\mathcal{U}_{ABA})_*(G')$ and id'_A are connected, we conclude that the 0-cells \tilde{G}' and G' are also connected. \square

Thus we proved the first part of Theorem 4.1.

We would like to conclude this subsection with the observation that mapping spaces of the simplicial category $\mathbf{HoAlg}_\mathcal{C}^\Delta$ enjoy the following remarkable property

Corollary 4.5 *Let A_1, A_2, A_3 be $\text{Cobar}(\mathcal{C})$ -algebras and F be an ∞ quasi-isomorphism from A_1 to A_2 . Then the composition (resp. pre-composition) with F induces the weak equivalences of simplicial sets:*

$$\begin{aligned}\mathfrak{MC}_\bullet(\mathbf{map}(A_3, A_1)) &\rightarrow \mathfrak{MC}_\bullet(\mathbf{map}(A_3, A_2)), \\ \mathfrak{MC}_\bullet(\mathbf{map}(A_2, A_3)) &\rightarrow \mathfrak{MC}_\bullet(\mathbf{map}(A_1, A_3)).\end{aligned}$$

Proof. The desired statement is a direct consequence of Proposition 4.3 and [2, Theorem 2.2]. \square

4.2 The functor \mathfrak{F} from Theorem 4.1 has the desired universal property

The proof of the universal property of the functor \mathfrak{F} is based on the following proposition:

Proposition 4.6 *Let $\mathfrak{G} : \text{Cat}_\mathcal{C} \rightarrow \mathfrak{D}$ be a functor which sends ∞ quasi-isomorphisms to isomorphisms in \mathfrak{D} . Let A, B be $\text{Cobar}(\mathcal{C})$ -algebras and F, G be ∞ -morphisms from A to B . If the corresponding 0-cells F' and G' of $\mathfrak{MC}_\bullet(\mathbf{map}(A, B))$ are connected then*

$$\mathfrak{G}(F) = \mathfrak{G}(G).$$

Proof. By the condition of the proposition, there exists a 1-cell

$$K' \in \text{Hom}(\mathcal{C}(A), B) \hat{\otimes} \mathbb{k}[t, dt] \tag{4.28}$$

such that

$$K' \Big|_{t=dt=0} = F' \tag{4.29}$$

and

$$K' \Big|_{t=1, dt=0} = G'. \tag{4.30}$$

Since

$$\mathcal{C}(A) = \bigoplus_n (\mathcal{C}(n) \otimes A^{\otimes n})_{S_n}$$

and $\text{Hom}(\mathcal{C}(A), B)$ is considered with the topology coming from filtration (2.14), we have the natural strict \mathfrak{SLie}_∞ -morphism

$$\text{Hom}(\mathcal{C}(A), B) \hat{\otimes} \mathbb{k}[t, dt] \rightarrow \text{Hom}(\mathcal{C}(A), B \otimes \mathbb{k}[t, dt]). \tag{4.31}$$

Therefore the 1-cell K' gives us an ∞ -morphism K from A to $B \otimes \mathbb{k}[t, dt]$ which fits into the following commutative diagram

$$\begin{array}{ccc}
 & & B \\
 & \nearrow F & \nearrow p_0 \\
 A & \xrightarrow{K} & B \otimes \mathbb{k}[t, dt] \\
 & \searrow G & \searrow p_1 \\
 & & B
 \end{array}
 \quad (4.32)$$

where $B \otimes \mathbb{k}[t, dt]$ is considered with the differential $\partial_B + dt \partial_t$ and the natural $\text{Cobar}(\mathcal{C})$ -structure coming from the one on B . Moreover, p_0 and p_1 are the obvious (strict) morphisms of $\text{Cobar}(\mathcal{C})$ -algebras

$$p_0(v) := v|_{t=dt=0} : B \otimes \mathbb{k}[t, dt] \rightarrow B \quad (4.33)$$

and

$$p_1(v) := v|_{t=1, dt=0} : B \otimes \mathbb{k}[t, dt] \rightarrow B. \quad (4.34)$$

Let us observe that the maps p_0 and p_1 fit into the commutative diagram

$$\begin{array}{ccc}
 & & B \\
 & \nearrow \text{id}_B & \nearrow p_0 \\
 B & \xrightarrow{i} & B \otimes \mathbb{k}[t, dt] \\
 & \searrow \text{id}_B & \searrow p_1 \\
 & & B
 \end{array}
 \quad (4.35)$$

where $i : B \rightarrow B \otimes \mathbb{k}[t, dt]$ is the natural embedding given by

$$i(v) := v \otimes 1.$$

Applying the functor \mathfrak{G} to (4.35), we get

$$\mathfrak{G}(p_0) \circ \mathfrak{G}(i) = \mathfrak{G}(p_1) \circ \mathfrak{G}(i) = \text{id}_{\mathfrak{G}(B)}.$$

Hence, since i is obviously a quasi-isomorphism, we deduce that

$$\mathfrak{G}(p_0) = \mathfrak{G}(p_1).$$

Finally, applying the functor \mathfrak{G} to (4.32), we get

$$\mathfrak{G}(p_0) \circ \mathfrak{G}(K) = \mathfrak{G}(F), \quad \mathfrak{G}(p_1) \circ \mathfrak{G}(K) = \mathfrak{G}(G)$$

which implies that $\mathfrak{G}(F) = \mathfrak{G}(G)$. □

Proposition 4.6 motivates the following definition⁸

⁸For several other justifications of this definition, we refer the reader to paper [8] by V. Dotsenko and N. Poncin.

Definition 4.7 Let A, B be $\text{Cobar}(\mathcal{C})$ -algebras. We say that ∞ -morphisms F, G from A to B are homotopic if the corresponding 0-cells F' and G' are connected in $\mathfrak{MC}_\bullet(\mathbf{map}(A, B))$.

We can now prove the universal property of functor (4.2).

Indeed, let \mathfrak{G} be a functor from $\text{Cat}_{\mathcal{C}}$ to some category \mathfrak{D} which sends ∞ quasi-isomorphisms to isomorphisms.

For objects of A, B, \dots of $\pi_0(\text{HoAlg}_{\mathcal{C}}^{\Delta})$ we set

$$\mathfrak{G}'(A) := \mathfrak{G}(A).$$

Next, given two $\text{Cobar}(\mathcal{C})$ -algebras A, B and an isomorphism class

$$[F'] \in \pi_0(\mathfrak{MC}_\bullet(\mathbf{map}(A, B)))$$

of a MC element F' in $\mathbf{map}(A, B)$ we set

$$\mathfrak{G}([F']) := \mathfrak{G}(F), \quad (4.36)$$

where F is the ∞ -morphism from A to B corresponding to the MC element F' .

Due to Proposition 4.6, the right hand side of (4.36) does not depend on the choice of the MC element F' in its isomorphism class $[F']$.

It is clear that, this way, we get a functor

$$\mathfrak{G}' : \pi_0(\text{HoAlg}_{\mathcal{C}}^{\Delta}) \rightarrow \mathfrak{D}$$

satisfying $\mathfrak{G}' \circ \mathfrak{F} = \mathfrak{G}$.

It is also clear that such a functor \mathfrak{G}' is unique and the proof of Theorem 4.1 is complete. We can use the above results to obtain the following converse to Cor. 4.5. Together they give a recognition principal for ∞ quasi-isomorphisms, in analogy with the characterization of weak equivalences via function complexes in a simplicial model category.

Corollary 4.8 For every ∞ -morphism $F : A \rightarrow B$ of $\text{Cobar}(\mathcal{C})$ algebras, the following statements are equivalent:

1. F is an ∞ -quasi-isomorphism.
2. The \mathfrak{GLie}_{∞} -morphisms

$$\mathcal{U}_{AAB} : \mathbf{map}(A, A) \rightarrow \mathbf{map}(A, B) \quad \text{and} \quad \mathcal{U}_{BAB} : \mathbf{map}(B, A) \rightarrow \mathbf{map}(B, B)$$

induce homotopy equivalences of simplicial sets

$$\mathfrak{MC}_\bullet(\mathbf{map}(A, A)) \xrightarrow{\sim} \mathfrak{MC}_\bullet(\mathbf{map}(A, B)) \quad \text{and} \quad \mathfrak{MC}_\bullet(\mathbf{map}(B, A)) \xrightarrow{\sim} \mathfrak{MC}_\bullet(\mathbf{map}(B, B))$$

respectively.

3. The \mathfrak{GLie}_{∞} -morphisms

$$\mathcal{U}_{ABA} : \mathbf{map}(B, A) \rightarrow \mathbf{map}(A, A) \quad \text{and} \quad \mathcal{U}_{ABB} : \mathbf{map}(B, B) \rightarrow \mathbf{map}(A, B)$$

induce homotopy equivalences of simplicial sets

$$\mathfrak{MC}_\bullet(\mathbf{map}(B, A)) \xrightarrow{\sim} \mathfrak{MC}_\bullet(\mathbf{map}(A, A)) \quad \text{and} \quad \mathfrak{MC}_\bullet(\mathbf{map}(B, B)) \xrightarrow{\sim} \mathfrak{MC}_\bullet(\mathbf{map}(A, B))$$

respectively.

Proof. The implications $1 \Rightarrow 2$ and $1 \Rightarrow 3$ are particular cases of Corollary 4.5. Let us prove the implication $2 \Rightarrow 1$.

Since F induces a homotopy equivalence of simplicial sets

$$\mathfrak{M}\mathfrak{C}_\bullet(\mathbf{map}(B, A)) \xrightarrow{\sim} \mathfrak{M}\mathfrak{C}_\bullet(\mathbf{map}(B, B)),$$

we have an isomorphism

$$\pi_0\left(\mathfrak{M}\mathfrak{C}_\bullet(\mathbf{map}(B, A))\right) \cong \pi_0\left(\mathfrak{M}\mathfrak{C}_\bullet(\mathbf{map}(B, B))\right),$$

which implies that there exists an ∞ -morphism $G : B \rightarrow A$ such that $F \circ G$ and id_B are homotopic, in the sense of Def. 4.7. As in the proof of Prop. 4.6, this gives a commutative diagram of $\text{Cobar}(\mathcal{C})$ algebras and ∞ -morphisms:

$$\begin{array}{ccc} & & B \\ & \nearrow^{F \circ G} & \\ B & \xrightarrow{H} & B \otimes \mathbb{k}[t, dt] \\ & \searrow_{\text{id}_B} & \\ & & B \end{array} \quad \begin{array}{c} p_0 \\ p_1 \end{array}$$

By taking the linear terms of the ∞ -morphisms, this diagram, in turn, gives a diagram of cochain complexes

$$\begin{array}{ccc} & & B \\ & \nearrow^{f \circ g} & \\ B & \xrightarrow{h} & B \otimes \mathbb{k}[t, dt] \\ & \searrow_{\text{id}_B} & \\ & & B, \end{array} \quad \begin{array}{c} p_0 \\ p_1 \end{array}$$

where h and $f \circ g$ are the linear terms $p_{B \otimes \mathbb{k}[t, dt]} \circ H|_B$, and $p_B \circ F \circ G|_B$, respectively. Let $I : B \otimes \mathbb{k}[t, dt] \rightarrow B$ denote “integration over the fiber”, i.e. the degree -1 linear map

$$I\left(b \otimes q(t) + \tilde{b} \otimes \tilde{q}(t)dt\right) = (-1)^{|\tilde{b}|} \tilde{b} \int_0^1 \tilde{q}(t)dt.$$

Then one can show that the map $s_B : B \rightarrow B$ defined as

$$s_B := I \circ h$$

is a chain homotopy

$$\text{id}_B - f \circ g = \partial_B s_B + s_B \partial_B.$$

Hence, the map f is surjective on cohomology.

Now consider the \mathfrak{SLie}_∞ -morphism

$$\mathcal{U}_{ABB} : \mathbf{map}(B, B) \rightarrow \mathbf{map}(A, B)$$

which sends a MC element K' to $(K \circ F)'$. This gives a map between sets

$$\pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(B, B))\right) \rightarrow \pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(A, B))\right).$$

Since the MC elements $(F \circ G)'$ and $(\mathrm{id}_B)'$ are connected by a 1-cell, we conclude that F' is connected to $(F \circ G \circ F)'$. By the first part of statement 2, the \mathfrak{SLie}_∞ -morphism

$$\mathcal{U}_{AAB} : \mathbf{map}(A, A) \rightarrow \mathbf{map}(A, B)$$

induces a bijection

$$\pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(A, A))\right) \cong \pi_0\left(\mathfrak{MC}_\bullet(\mathbf{map}(A, B))\right).$$

sending $[(\mathrm{id}_A)']$ to $[F']$, and $[(G \circ F)']$ to $[(F \circ G \circ F)'] = [F']$. Hence, we see that id_A is homotopic to $G \circ F$. We produce a chain homotopy $s_A : A \rightarrow A$ using the same construction as before, which shows that f is also injective on cohomology. Hence, $F : A \rightarrow B$ is an ∞ -quasi-isomorphism.

The proof of the implication $3 \Rightarrow 1$ is very similar to that of $2 \Rightarrow 1$. So we leave it to the reader. \square

5 The Homotopy Transfer Theorem is a simple consequence of the Goldman-Millson theorem

In this section we give an elegant proof of the Homotopy Transfer Theorem for $\mathrm{Cobar}(\mathcal{C})$ -algebras. This proof is based on a construction from [7] and a version of the Goldman-Millson theorem from [2].

Let us consider a dg cooperad \mathcal{C} for which \mathcal{C}_\circ carries ascending filtration (2.1) satisfying condition (2.3) and let A, B be cochain complexes.

In [7, Section 3.1], we equipped the cochain complex

$$\mathrm{Cyl}(\mathcal{C}, A, B) := \mathbf{s}^{-1} \mathrm{Hom}(\mathcal{C}_\circ(A), A) \oplus \mathrm{Hom}(\mathcal{C}(A), B) \oplus \mathbf{s}^{-1} \mathrm{Hom}(\mathcal{C}_\circ(B), B) \quad (5.1)$$

with a natural \mathfrak{SLie}_∞ -algebra structure⁹

According to [7, Section 3.1], MC elements of $\mathrm{Cyl}(\mathcal{C}, A, B)$ are triples:

- a $\mathrm{Cobar}(\mathcal{C})$ -algebra structure on A ,
- a $\mathrm{Cobar}(\mathcal{C})$ -algebra structure on B , and
- an ∞ -morphism from A to B .

⁹In [7], we actually introduce an L_∞ -structure on the suspension of (5.1). But the latter is, of course, equivalent to introducing a \mathfrak{SLie}_∞ -algebra structure on (5.1).

In particular, any chain map $\varphi : A \rightarrow B$ gives a MC element Q_φ corresponding to the zero $\text{Cobar}(\mathcal{C})$ -algebra structures on A , B , and the strict ∞ -morphism from A to B .

Let us twist the $\mathfrak{S}\text{Lie}_\infty$ -algebra structure on (5.1), and denote the new $\mathfrak{S}\text{Lie}_\infty$ -algebra by

$$\text{Cyl}(\mathcal{C}, A, B)^{Q_\varphi}. \quad (5.2)$$

It is not hard to see that the graded subspace

$$\text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi} := \mathfrak{s}^{-1} \text{Hom}(\mathcal{C}_\circ(A), A) \oplus \text{Hom}(\mathcal{C}_\circ(A), B) \oplus \mathfrak{s}^{-1} \text{Hom}(\mathcal{C}_\circ(B), B) \quad (5.3)$$

is a $\mathfrak{S}\text{Lie}_\infty$ -subalgebra of (5.2) and filtration (2.1) on \mathcal{C}_\circ allows us to equip (5.3) with a natural complete descending filtration $\mathcal{F}_\bullet \text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi}$ (see Remark 2 in [7, Section 3.2]) such that

$$\text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi} = \mathcal{F}_1 \text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi}. \quad (5.4)$$

In other words, (5.3) is a filtered $\mathfrak{S}\text{Lie}_\infty$ -algebra.

Furthermore, according to [7, Section 3.2], MC elements of (5.3) are in bijection with triples:

- a $\text{Cobar}(\mathcal{C})$ -algebra structure on A ,
- a $\text{Cobar}(\mathcal{C})$ -algebra structure on B , and
- an ∞ -morphism F from A to B whose linear term is φ .

The Homotopy Transfer Theorem can be now formulated as follows:

Theorem 5.1 *Let B be a $\text{Cobar}(\mathcal{C})$ -algebra, A be a cochain complex and $\varphi : A \rightarrow B$ be a quasi-isomorphism of cochain complexes. Then there exists a $\text{Cobar}(\mathcal{C})$ -algebra structure Q_A on A , a $\text{Cobar}(\mathcal{C})$ -algebra structure Q_B on B (which is homotopy equivalent to the original one) and an ∞ -morphism F from (A, Q_A) to (B, Q_B) whose linear term is φ . If $(\tilde{Q}_A, \tilde{Q}_B, \tilde{F})$ is another triple satisfying the above properties then the MC elements corresponding to (Q_A, Q_B, F) and $(\tilde{Q}_A, \tilde{Q}_B, \tilde{F})$ are isomorphic in*

$$\mathfrak{MC}_\bullet(\text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi}).$$

Proof. Let us identify the graded vector space of the convolution Lie algebra

$$\text{Conv}(\mathcal{C}_\circ, \text{End}_B)$$

with $\text{Hom}(\mathcal{C}_\circ(B), B)$ and consider $\mathfrak{s}^{-1} \text{Hom}(\mathcal{C}_\circ(B), B)$ with the corresponding $\mathfrak{S}\text{Lie}_\infty$ -algebra structure.

Due to [7, Proposition 3.2], the canonical projection

$$\pi_B : \text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi} \rightarrow \mathfrak{s}^{-1} \text{Hom}(\mathcal{C}_\circ(B), B) \quad (5.5)$$

is a strict quasi-isomorphism of $\mathfrak{S}\text{Lie}_\infty$ -algebras which is obviously compatible with the descending filtrations coming from (2.1).

Using the same arguments, as in the proof of [7, Proposition 3.2], it is easy to see that

$$\pi_B : \mathcal{F}_m \text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi} \rightarrow \mathbf{s}^{-1} \mathcal{F}_m \text{Hom}(\mathcal{C}_\circ(B), B)$$

is a quasi-isomorphism of cochain complexes for every $m \geq 1$.

Therefore, applying Theorem 1.1 from [2] to (5.5), we conclude that π_B induces a weak equivalence of simplicial sets

$$\mathfrak{MC}_\bullet(\text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi}) \rightarrow \mathfrak{MC}_\bullet(\mathbf{s}^{-1} \text{Hom}(\mathcal{C}_\circ(B), B))$$

and hence a bijection

$$\pi_0\left(\mathfrak{MC}_\bullet(\text{Cyl}_\circ(\mathcal{C}, A, B)^{Q_\varphi})\right) \rightarrow \pi_0\left(\mathfrak{MC}_\bullet(\mathbf{s}^{-1} \text{Hom}(\mathcal{C}_\circ(B), B))\right).$$

Thus Theorem 5.1 is a simple consequence of the fact that MC elements of the \mathfrak{SLie}_∞ -algebra

$$\mathbf{s}^{-1} \text{Hom}(\mathcal{C}_\circ(B), B) \tag{5.6}$$

are in bijection with $\text{Cobar}(\mathcal{C})$ -algebra structures on B . Moreover, homotopy equivalent $\text{Cobar}(\mathcal{C})$ -algebra structures on B correspond precisely to isomorphic MC elements of (5.6). \square

Remark 5.2 In the usual version of the Homotopy Transfer Theorem (see [19, Theorem 10.3.2]) one constructs a $\text{Cobar}(\mathcal{C})$ -algebra structure on A and an ∞ quasi-isomorphism F from A to B with the original $\text{Cobar}(\mathcal{C})$ -algebra structure, while in the above theorem, F lands in (B, Q_B) where Q_B is only homotopy equivalent to the original one. On the other hand, given an isomorphism connecting two MC elements \tilde{Q}_B and Q_B in

$$\mathfrak{MC}_\bullet(\mathbf{s}^{-1} \text{Hom}(\mathcal{C}_\circ(B), B)),$$

it is easy to construct an ∞ -morphism G from (B, Q_B) to (B, \tilde{Q}_B) whose linear term is id_B . So the “usual” version of the Homotopy Transfer Theorem follows from Theorem 5.1.

A Proof of Proposition 2.6

Although very similar claims to Proposition 2.6 appeared in the literature (see, for example, [1], [12], [16], [22]) we still decided to give its proof for convenience of the reader.

Let us denote by Q'_A the composition

$$p_A \circ Q_A : \mathcal{C}(A) \rightarrow A.$$

To prove that multi-bracket (2.12) is symmetric in its arguments, we let

$$\Delta_m(v) = \sum_{\alpha} (\gamma_{\alpha}; v_1^{\alpha}, v_2^{\alpha}, \dots, v_m^{\alpha}) \tag{A.1}$$

and recall that $\Delta_m(v)$ lands in S_m -invariants of $\mathcal{C}(m) \otimes V^{\otimes m}$. In other words, for every $\sigma \in S_m$ we have

$$\sum_{\alpha} (\sigma^{-1}(\gamma_{\alpha}); v_{\sigma(1)}^{\alpha}, v_{\sigma(2)}^{\alpha}, \dots, v_{\sigma(m)}^{\alpha}) = \sum_{\alpha} (\gamma_{\alpha}; v_1^{\alpha}, v_2^{\alpha}, \dots, v_m^{\alpha}). \quad (\text{A.2})$$

Therefore, for every $\sigma \in S_m$, we have

$$\begin{aligned} \{f_{\sigma(1)}, \dots, f_{\sigma(m)}\}(v) &= \\ \sum_{\alpha} Q'_A \circ (1 \otimes f_{\sigma(1)} \otimes \dots \otimes f_{\sigma(m)}) (\sigma^{-1}(\gamma_{\alpha}); v_{\sigma(1)}^{\alpha}, v_{\sigma(2)}^{\alpha}, \dots, v_{\sigma(m)}^{\alpha}) &= \\ \sum_{\alpha} \pm Q'_A (\sigma^{-1}(\gamma_{\alpha}); f_{\sigma(1)}(v_{\sigma(1)}^{\alpha}), f_{\sigma(2)}(v_{\sigma(2)}^{\alpha}), \dots, f_{\sigma(m)}(v_{\sigma(m)}^{\alpha})) & . \end{aligned}$$

Thus, since Q_A is compatible with the action of the symmetric group,

$$\begin{aligned} \{f_{\sigma(1)}, \dots, f_{\sigma(m)}\}(v) &= \\ \sum_{\alpha} \pm Q'_A (\sigma^{-1}(\gamma_{\alpha}); f_{\sigma(1)}(v_{\sigma(1)}^{\alpha}), f_{\sigma(2)}(v_{\sigma(2)}^{\alpha}), \dots, f_{\sigma(m)}(v_{\sigma(m)}^{\alpha})) &= \\ \sum_{\alpha} \pm Q'_A (\gamma_{\alpha}; f_1(v_1^{\alpha}), f_2(v_2^{\alpha}), \dots, f_m(v_m^{\alpha})) &= \\ \sum_{\alpha} \pm Q'_A \circ (1 \otimes f_1 \otimes \dots \otimes f_m) (\gamma_{\alpha}; v_1^{\alpha}, v_2^{\alpha}, \dots, v_m^{\alpha}) &= \pm \{f_1, \dots, f_m\}(v). \end{aligned}$$

So multi-bracket (2.12) is indeed symmetric¹⁰ in its arguments.

Let us now prove that the operation

$$f \mapsto \{f\} \quad (\text{A.3})$$

is a differential on $\text{Hom}(V, A)$, i.e. $\{\{f\}\} = 0$ for every $f \in \text{Hom}(V, A)$.

Indeed, using the identities $\partial_A^2 = 0$, $\partial_V^2 = 0$ and the compatibility of Δ_1 with the differentials on V and $\mathcal{C}(1) \otimes V$, we get

$$\begin{aligned} \{\{f\}\}(v) &= \partial_A \{f\}(v) - (-1)^{|f|+1} \{f\}(\partial_V v) + Q'_A (1 \otimes \{f\}(\Delta_1(v))) = \\ &= \partial_A \circ Q'_A \circ (1 \otimes f) \circ \Delta_1(v) + (-1)^{|f|} Q'_A \circ (1 \otimes f) ((\partial_C + \partial_V) \Delta_1(v)) \\ &+ Q'_A \circ (1 \otimes (\partial_A \circ f)) (\Delta_1(v)) - (-1)^{|f|} Q'_A \circ (1 \otimes (f \circ \partial_V)) (\Delta_1(v)) \\ &+ Q'_A \circ (1 \otimes Q'_A) \circ (1 \otimes 1 \otimes f) ((1 \otimes \Delta_1) \circ \Delta_1(v)) \quad (\text{A.4}) \end{aligned}$$

for every $v \in V$.

Using the axioms of a coalgebra over a cooperad, we rewrite the expression $(1 \otimes \Delta_1) \circ \Delta_1(v)$ in (A.4) as follows

$$(1 \otimes \Delta_1) \circ \Delta_1(v) = (\Delta^c \otimes 1) \circ \Delta_1(v), \quad (\text{A.5})$$

¹⁰In the above calculations, the sign factors can be easily deduced from the Koszul rule of signs.

where $\Delta^{\mathcal{C}}$ is the cooperadic comultiplication

$$\Delta^{\mathcal{C}} : \mathcal{C}(1) \rightarrow \mathcal{C}(1) \otimes \mathcal{C}(1).$$

Therefore the expression $\{\{f\}\}(v)$ can be rewritten as

$$\{\{f\}\}(v) = (\partial_A \circ Q'_A + Q'_A \circ (\partial_{\mathcal{C}} + \partial_A) + Q'_A \circ Q_A) \circ (1 \otimes f) \circ \Delta_1(v). \quad (\text{A.6})$$

Thus the identity $\{\{f\}\} = 0$ is a consequence of the MC equation for Q'_A .

Our goal now is to prove the relation

$$\sum_{p=1}^m \sum_{\sigma \in \text{Sh}_{p, m-p}} (-1)^{\varepsilon(\sigma; f_1, \dots, f_m)} \{\{f_{\sigma(1)}, \dots, f_{\sigma(p)}\}, f_{\sigma(p+1)}, \dots, f_{\sigma(m)}\}(v) = 0 \quad (\text{A.7})$$

for every $m \geq 2$ and $v \in V$, where the sign factor $(-1)^{\varepsilon(\sigma; f_1, \dots, f_m)}$ is defined in (1.1).

In our calculations below, we often put \pm instead of the precise sign factor. These sign factors can be easily deduced from the Koszul rule of signs.

Unfolding (A.7) we get

$$\begin{aligned} & \sum_{p=1}^m \sum_{\sigma \in \text{Sh}_{p, m-p}} (-1)^{\varepsilon(\sigma; f_1, \dots, f_m)} \{\{f_{\sigma(1)}, \dots, f_{\sigma(p)}\}, f_{\sigma(p+1)}, \dots, f_{\sigma(m)}\}(v) = \\ & \quad \partial_A(\{f_1, \dots, f_m\}(v)) + (-1)^{|f_1| + \dots + |f_m|} \{f_1, \dots, f_m\}(\partial_V(v)) \\ & \quad + \sum_{i=1}^m (-1)^{|f_1| + \dots + |f_{i-1}|} \{f_1, \dots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \dots, f_m\}(v) \\ & + \sum_{\substack{1 \leq p \leq m \\ \sigma \in \text{Sh}_{p, m-p}}} (-1)^{\varepsilon(\sigma; f_1, \dots, f_m)} Q'_A(1; Q'_A((1; f_{\sigma(1)}, \dots, f_{\sigma(p)}) \Delta_p(-)), f_{\sigma(p+1)}, \dots, f_{\sigma(m)}(\Delta_{m-p+1}(v))). \end{aligned} \quad (\text{A.8})$$

Expanding the expression $\partial_A(\{f_1, \dots, f_m\}(v))$, we get

$$\begin{aligned} \partial_A(\{f_1, \dots, f_m\}(v)) &= \partial_A Q'_A(1; f_1, \dots, f_m(\Delta_m(v))) \\ &= \sum_{\alpha} \partial_A Q'_A((1; f_1, \dots, f_m)(\gamma_{\alpha}; v_1^{\alpha}, v_2^{\alpha}, \dots, v_m^{\alpha})) \\ &= \sum_{\alpha} \pm \partial_A Q'_A(\gamma_{\alpha}; f_1(v_1^{\alpha}), \dots, f_m(v_m^{\alpha})). \end{aligned}$$

Using

$$\Delta_m(\partial_V(v)) = \sum_{\alpha} (\partial_{\mathcal{C}}(\gamma_{\alpha}); v_1^{\alpha}, v_2^{\alpha}, \dots, v_m^{\alpha}) + \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm (\gamma_{\alpha}; v_1^{\alpha}, \dots, \partial_V(v_i^{\alpha}), \dots, v_m^{\alpha}),$$

we expand $\{f_1, \dots, f_m\}(\partial_V(v))$ obtaining

$$\begin{aligned}
\{f_1, \dots, f_m\}(\partial_V(v)) &= Q'_A(1; f_1, \dots, f_m(\Delta_m(\partial_V(v)))) \\
&= \sum_{\alpha} Q'_A((1; f_1, \dots, f_m)(\partial_{\mathcal{C}}(\gamma_{\alpha}); v_1^{\alpha}, v_2^{\alpha}, \dots, v_m^{\alpha})) \\
&+ \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A((1; f_1, \dots, f_m)(\gamma_{\alpha}; v_1^{\alpha}, \dots, \partial_V(v_i^{\alpha}), \dots, v_m^{\alpha})) \\
&= \sum_{\alpha} \pm Q'_A(\partial_{\mathcal{C}}(\gamma_{\alpha}); f_1(v_1^{\alpha}), \dots, f_m(v_m^{\alpha})) \\
&+ \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(\gamma_{\alpha}; f_1(v_1^{\alpha}), \dots, f_i(\partial_V(v_i^{\alpha})), \dots, f_m(v_m^{\alpha})).
\end{aligned}$$

Expanding the sum

$$\sum_{i=1}^m (-1)^{|f_1|+\dots+|f_{i-1}|} \{f_1, \dots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \dots, f_m\}(v)$$

we obtain

$$\begin{aligned}
\sum_{i=1}^m (-1)^{|f_1|+\dots+|f_{i-1}|} \{f_1, \dots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \dots, f_m\}(v) &= \\
\sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(1; f_1, \dots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V), \dots, f_m)(\gamma_{\alpha}; v_1^{\alpha}, v_2^{\alpha}, \dots, v_m^{\alpha}) & \\
= \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(\gamma_{\alpha}; f_1(v_1^{\alpha}), \dots, (\partial_A \circ f_i - (-1)^{|f_i|} f_i \circ \partial_V)(v_i^{\alpha}), \dots, f_m(v_m^{\alpha})) & \\
= \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(\gamma_{\alpha}; f_1(v_1^{\alpha}), \dots, \partial_A(f_i(v_i^{\alpha})), \dots, f_m(v_m^{\alpha})) & \\
- \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(\gamma_{\alpha}; f_1(v_1^{\alpha}), \dots, f_i(\partial_V v_i^{\alpha}), \dots, f_m(v_m^{\alpha})). & \quad (\text{A.9})
\end{aligned}$$

The last sum in the R.H.S. of (A.8) is expanded as follows:

$$\begin{aligned}
\sum_{\substack{1 \leq p \leq m \\ \sigma \in \text{Sh}_{p, m-p}}} (-1)^{\varepsilon(\sigma; f_1, \dots, f_m)} Q'_A(1; Q'_A((1; f_{\sigma(1)}, \dots, f_{\sigma(p)}) \Delta_p(-)), f_{\sigma(p+1)}, \dots, f_{\sigma(m)}(\Delta_{m-p+1}(v))) & \\
= \sum_{\substack{1 \leq p \leq m \\ \sigma \in \text{Sh}_{p, m-p}}} \pm Q'_A(1 \otimes Q'_A \otimes 1^{\otimes(m-p)})(1; (1; f_{\sigma(1)}, \dots, f_{\sigma(p)}), f_{\sigma(p+1)}, \dots, f_{\sigma(m)}) & \\
((1 \otimes \Delta_p \otimes 1^{\otimes(m-p)}) \circ \Delta_{m-p+1}(v)). & \quad (\text{A.10})
\end{aligned}$$

Using the axioms of \mathcal{C} -coalgebra structure on V , we can rewrite the term $(1 \otimes \Delta_p \otimes 1^{\otimes(m-p)}) \circ \Delta_{m-p+1}(v)$ as follows:

$$(1 \otimes \Delta_p \otimes 1^{\otimes(m-p)}) \circ \Delta_{m-p+1}(v) = (\Delta_{\mathbf{t}_{m,p}}^{\mathcal{C}} \otimes 1^{\otimes m}) \circ \Delta_m(v), \quad (\text{A.11})$$

where $\mathbf{t}_{m,p}$ is the labeled planar tree depicted in Figure 3. and let $\gamma_{\alpha,\beta,1}^{\sigma}$ and $\gamma_{\alpha,\beta,2}^{\sigma}$ be the

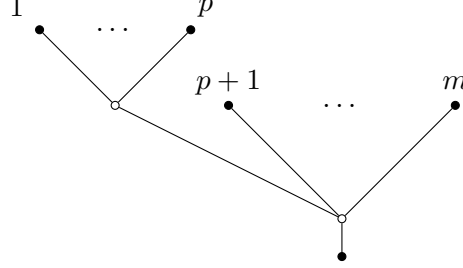


Figure 3: The labeled planar tree $\mathbf{t}_{m,p}$

tensor factors in

$$\Delta_{\sigma(\mathbf{t}_{m,p})}^{\mathcal{C}}(\gamma_{\alpha}) = \sum_{\beta} \gamma_{\alpha,\beta,1}^{\sigma} \otimes \gamma_{\alpha,\beta,2}^{\sigma}, \quad (\text{A.12})$$

where $\sigma(\mathbf{t}_{m,p})$ is the tree corresponding to the $(p, m-p)$ -shuffle σ .

Using the axioms of the cooperadic comultiplication $\Delta^{\mathcal{C}}$, we get

$$\text{Sum (A.10)} =$$

$$\sum_{\substack{\alpha,\beta \\ 1 \leq p \leq m \\ \sigma \in \text{Sh}_{p,m-p}}} \pm Q'_A(1 \otimes Q'_A \otimes 1^{\otimes(m-p)}) \quad (\text{A.13})$$

$$\left(\gamma_{\alpha,\beta,1}^{\sigma}; (\gamma_{\alpha,\beta,2}^{\sigma}; f_{\sigma(1)}(v_{\sigma(1)}^{\alpha}), \dots, f_{\sigma(p)}(v_{\sigma(p)}^{\alpha})), f_{\sigma(p+1)}(v_{\sigma(p+1)}^{\alpha}), \dots, f_{\sigma(m)}(v_{\sigma(m)}^{\alpha}) \right),$$

which can be rewritten in terms of the bracket on the convolution Lie algebra $\text{Conv}(\mathcal{C}_o, \text{End}_A)$ (Prop. 4.1, [3]). Namely,

$$\text{Sum (A.10)} =$$

$$\frac{1}{2} \sum_{\alpha} [Q'_A, Q'_A](\gamma_{\alpha}; f_1(v_1^{\alpha}), \dots, f_m(v_m^{\alpha})), \quad (\text{A.14})$$

where we tacitly identify the composition $Q'_A = p_A \circ Q_A$ with the corresponding element in

$$\text{Conv}(\mathcal{C}_o, \text{End}_A) = \prod_{n \geq 1} \text{Hom}_{S_n}(\mathcal{C}_o(n), \text{End}_A(n)). \quad (\text{A.15})$$

Collecting the expanded terms of the R.H.S. of (A.8) we obtain

$$\sum_{\alpha} \pm \partial_A Q'_A(\gamma_{\alpha}; f_1(v_1^{\alpha}), \dots, f_m(v_m^{\alpha})) + \sum_{\alpha} \pm Q'_A(\partial_{\mathcal{C}}(\gamma_{\alpha}); f_1(v_1^{\alpha}), \dots, f_m(v_m^{\alpha}))$$

$$+ \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \dots, f_i(\partial_V(v_i^\alpha)), \dots, f_m(v_m^\alpha)) \quad (\text{A.16})$$

$$- \sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \dots, f_i(\partial_V v_i^\alpha), \dots, f_m(v_m^\alpha)) \quad (\text{A.17})$$

$$\sum_{\substack{\alpha \\ 1 \leq i \leq m}} \pm Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \dots, \partial_A(f_i(v_i^\alpha)), \dots, f_m(v_m^\alpha)) + \frac{1}{2} \sum_{\alpha} [Q'_A, Q'_A](\gamma_\alpha; f_1(v_1^\alpha), \dots, f_m(v_m^\alpha))$$

Canceling terms (A.16) and (A.17), we see that the right hand side of (A.8) can be rewritten as

The R.H.S. of (A.8) =

$$\left(\partial_A \circ Q'_A + Q'_A \circ (\partial_C + \partial_A) + \frac{1}{2} [Q'_A, Q'_A] \right) (1; f_1, \dots, f_m)(\Delta_m(v)), \quad (\text{A.18})$$

where, as above, we identify Q'_A with the corresponding element in (A.15).

Thus desired equations (A.7) are satisfied due to the fact that Q'_A is a MC element of dg Lie algebra (A.15).

We conclude this proof by a comment about the sign factors.

All sign factors in the above computations are subject to the usual Koszul rule. For example, to show explicitly that terms (A.16) and (A.17) cancel each other, we need to verify that the corresponding contributions from the term

$$(-1)^{|f_1| + \dots + |f_m|} \{f_1, \dots, f_m\}(\partial_V(v)) \quad (\text{A.19})$$

matches with the contributions from

$$(-1)^{|f_1| + \dots + |f_i|} \{f_1, \dots, f_i \circ \partial_V, \dots, f_m\}(v). \quad (\text{A.20})$$

in equation (A.8).

It is easy to see that the contribution

$$Q'_A(\gamma_\alpha; f_1(v_1^\alpha), \dots, f_i \circ \partial_V(v_i^\alpha), \dots, f_m(v_m^\alpha)) \quad (\text{A.21})$$

from (A.19) and from (A.20) has the same sign factor

$$(-1)^{\varepsilon + \varepsilon'},$$

where

$$\varepsilon = |\gamma_\alpha|(1 + |f_1| + \dots + |f_m|) + |v_1^\alpha| + \dots + |v_{i-1}^\alpha| + |f_1| + \dots + |f_i|,$$

and

$$\varepsilon' = |v_1^\alpha|(|f_2| + \dots + |f_m|) + \dots + |v_{m-1}^\alpha||f_m|.$$

Proposition 2.6 is proved. \square

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